ZIMBABWE

Ministry of Primary and Secondary Education

Mathematics Module Level II Volume 2

Lifelong and Continuing Education

2020

Introduction

The last thirty years have seen resurgence in Open Distance learning as a pedagogical approach and this trend is envisaged to continue. The knowledge-based society that we live in has enabled learning to take place anywhere or everywhere. The concept of a classroom without walls continues to grow in Zimbabwe. Due to the demand for open distance learning, the Ministry of Primary and Secondary Education has revamped its non-formal education department to embed distance learning as a tool for learning in order to address the learning needs of the growing numbers of out of school learners or school drop outs that cannot access formal education systems. The module is written in a simple manner with lots of friendly and interactive activities to make learning interesting and easier for the out of school learners. The module develops critical thinking skills, problem solving skills among other 21st Century skills.

It is the Ministry's hope that out of school learners are going to take advantage of this module and benefit immensely in advancing their learning endeavours.

Acknowledgements

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We also thank Dr Lovemore Ndlovu, the Consultant in the Open Distance Learning Project.

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How to use this module

As you start this journey of acquiring a qualification in Ordinary Level Mathematics through open distance learning, it is critical that you understand the need to manage your study time and balance it with your day-to-day activities. This module will provide you with the basic material to assist you towards your public examinations in Mathematics.

This module has been subdivided into two volumes, that is, Volume 1 Volume 2. You are advised to study Volume 1 first before going to Volume 2.

Wish you the best!

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UNIT 16 - ALGEBRA 4 (Inequalities and Linear Programming)

CONTENTS

16.1 INTRODUCTION

What does the word inequality mean to you?

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The term inequality implies 'not equal'. Unequal objects might mean one is greater than the other or its less than the other. A lot of meaning can be deduced from unequal objects. Say you have a and b, we can have the following results;

1. $a < b$
2. $a \leq b$
3. $a > b$
4. $a \geq b$

In this unit we will look into solving inequalities, representing them on graphs and applying knowledge on inequalities in solving real life situations - linear programming.

OBJECTIVES

After going through the unit, you should be able to;

- ✓ solve simple linear inequalities,
- ✓ solve simultaneous inequalities

- ✓ represent inequalities on ;
    - number line
    - Cartesian plane
- ✓ translate verbal constrains into linear inequalities
- ✓ solve linear programming problems
- ✓ find the maximum and the minimum of a linear programming problem

KEY TERMS

≤ – means less than or equal to

≥ – means greater than or equal to

Integers – these are positive and negative whole numbers.

TIME

You should be done with this unit in 10 hours

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems on Inequalities and Linear Programming as possible for you to grasp all the concepts in this topic. Revisit the units on Algebra before undertaking this unit if you had not understood concepts on how to solve equations

Tip; in this unit, we are dealing with integers for the values of y and x

16.2 SOLVING INEQUALITIES

Inequalities are solved the same way we do equations. In some of the previous units we covered, we learnt how to solve equations. Do you still remember how to solved them? It is advisable that you revisit the previous units you learnt on Algebra in order to easily grasp concepts in this unit.

There are certain rules to be followed when solving inequalities, which are as follows;

1. Any number can be added or subtracted to both sides without affecting the inequality sign
2. Both sides of the inequality can be multiplied or divided by any positive value without affecting the inequality sign
3. If both sides of the inequality are multiplied or divided by a negative number the inequality sign must be reversed.

Let us start with solving simple linear inequalities

16.2.1 Solving simple linear inequalities

A linear inequality is an inequality with the highest power of the variable equal to 1. For example,

$$2x - 6 < 10$$
$$3 + 5y \geq 4$$

Let us consider an example in which we add both sides

Worked Example [1]

Question

Solve $x - 3 \geq 4$

Solution

$x - 3 \geq 4$ $x - 3 + 3 \geq 4 + 3$ $x + 0 \geq 7$ $x \geq 7$	$x - 3 \geq 4$ (taking -3 to the other side of the inequality sing it becomes $+3$ $x \geq 4 + 3$ $x \geq 7$

Let us consider the following example in which we subtract both sides

Worked Example [2]

Question

Solve $x + 6 < 10$

Solution

$x + 6 < 10$ $x + 6 - 6 < 10 - 6$ $x + 0 < 4$ $x < 4$	$x + 6 < 10$ (taking 6 to the other side of the sign , it becomes -6) $x < 10 - 6$ $x < 4$

Do this yourself;

Solve $x - 6 \leq 2$ (use the space below)

Solve $0 > x + 2$ (use the space below)

Let us now consider an example in which we divide both sides with a positive value (remember the inequality sign does not change or reverse)

Worked Example [3]

Question

Solve $5x \geq -5$

Solution

Here we are going to simply divide both sides by 5

$5x \geq -5$

$\frac{5x}{5} \geq \frac{-5}{5}$

$x \geq -1$

Do this yourself;

In the space below, solve $3x < x + 4$

Collect like terms to one side of the inequality sign

Now having done this lets consider an example in which we divide with a negative number

Tip; Watch out for sign change

Worked Example [4]

Question

Solve $-6x < 12$

Solution

You can use one of the two methods below

Divide both sides by a negative $-6x < 12$ $\frac{-6x}{-6} < \frac{12}{-6}$ $x > -2$ (we have reversed the sign)	Avoid dividing by a negative, interchange $-6x < 12$ Move $-6x$ to the other side and it will be a $+6x$ and 12 becomes -12 $-12 < 6x$ (now we divide by 6)

	$\frac{-12}{6} < \frac{6x}{6}$ $-2 < x$

These two answers are the same

Do this yourself;

In the space below, solve 5 – 2x > 9

Collect like terms

You could have collected like terms in one of the following ways

$5 - 9 > 2x$	$-2x > 9 - 5$

Finish off working out the solutions for both methods below

If all this have been done correctly your answer should be $x < -2$ or to write $-2 > x$

Let us now consider an example we will multiply with a positive number.

Remember, no sign change when dealing with positive numbers

Worked Example [5]

Question

Solve $\frac{x}{2} \geq 4$

Solution

Now we multiply both sides by 2

$\frac{x}{2} \geq 4$

$2(\frac{x}{2}) \geq 4(2)$

$x \geq 8$

Do this yourself;

In the space below, solve $\frac{x}{4} + 2 \leq 5$

Collect the like terms

Having collected like terms, you should obtain $\frac{x}{4} \leq 3$

Now you multiply both sides by 4 in the space below,

If all is done correctly you should obtain $x \leq 12$

Let us now consider an example where we multiply by a negative number

Remember when we multiply by a negative number we reverse the inequality sign

Worked Example [6]

Question

Solve $\frac{x}{-4} > 5$

Solution

You can use one of the two methods below

Multiplying by a negative $\frac{x}{-4} > 5$ $-4(\frac{x}{-4}) > 5\ (-4)$ reverse the inequality sign $x < -20$	Or avoid multiplying with the negative and change side change sign $\frac{x}{-4} > 5$ If you change sides it be like $-5 > \frac{x}{4}$ Multiply both sides by a 4 $-5(4) > \frac{x}{4}(4)$ $-20 > x$

Both the answer mean the same thing

Do this yourself;

Solve $6 - \frac{x}{3} < 0$

Do your working on the space provided using the two methods shown above

Taking 6 to the other side of the inequality sign	Taking $-\frac{x}{3}$ to the other side of the inequality sign

If all is done correctly your answer should be $x > 18$ or $18 < x$ these two statements mean the same thing

Now let us consider examples which involves multiple operations

Worked Example [7]

Question

Solve $6 - 4x < 2x + 2$

Solution

You have to collect like terms

$6 - 4x < 2x + 2$

$6 - 2 < 2x + 4x$

$4 < 6x$

Now we have to divide both sides by 6

$\frac{4}{6} < \frac{6x}{6}$

$\frac{4}{6} < x$ (reducing)

$\frac{2}{3} < x$

Do this yourself;

Solve $5(x - 4) > 2(2x + 11)$

Remove brackets and collect like terms (use the space below)

If all is done correctly you should obtain $x > 44$

Activity (*16.1*) Solving inequalities

Questions

Solve the following inequalities

1. $2x - 3 \leq 6$
2. $-5x > 25$
3. $\frac{1-11x}{4} \leq 3$
4. $2x + 7 < 1 - x$
5. $\frac{x}{6} - 2 > \frac{2x}{6}$

Answers

1) $x \leq 4.5$
2) $x < -5$
3) $x > -1$
4) $x < -2$
5) $x < -12$

16.2.2 Solving simultaneous inequalities

Do you know what simultaneous inequalities are?

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These are inequalities which have two inequality signs on one mathematical statement. Below are examples of simultaneous inequalities

$-4 \leq 2x + 4 \leq 12$

$2x - 2 \leq 3x < 5x + 4$

In the space below, write two examples of simultaneous inequalities

Let us consider examples on how we solve simultaneous inequalities

Worked Example [8]

Question

Solve $-4 \leq 2x + 4 \leq 12$ expressing your answer in the form $a \leq x \leq b$

Solution

We separate the inequality into two as follows

$-4 \leq 2x + 4$ $-4 \leq 2x + 4$ $-4 - 4 \leq 2x$ $-8 \leq 2x$ (we divide by 2) $\frac{-8}{2} \leq \frac{2x}{2}$ $-4 \leq x$	$2x + 4 \leq 12$ $2x \leq 12 - 4$ (collecting like terms) $2x \leq 8$ (we divide by 2) $\frac{2x}{2} \leq \frac{8}{2}$ $x \leq 4$

We then have to express our solution in the form '$a \leq x \leq b$'

$-4 \leq x \leq 4$

Let us look at another example of solving simultaneous inequalities

Worked Example [9]

Question

Solve $5x - 2 \leq 3x < 5x - 8$

Solution

Tip form two separate inequalities from the given simultaneous inequality

$5x - 2 \leq 3x < 5x - 8$

$5x - 2 \leq 3x$ Collect like terms $5x - 3x \leq 2$ $2x \leq 2$ We divide both sides by 2 $\frac{2x}{2} \leq \frac{2}{2}$ $x \leq 1$	$3x < 5x - 8$ collect like terms $8 < 5x - 3x$ $8 < 2x$ we divide both sides by 2 $\frac{8}{2} < \frac{2x}{2}$ $4 < x$

This math cannot be put in the form '$a \leq x < b$'

The solution here is in the form '$x \leq a$ or $x > b$'

Therefore the solution to this simultaneous inequality would be

$$x \leq 1 \text{ or } x > 4$$

Do this yourself;

Solve $x - 5 \leq 2x + 1 < 10 - x$

Separate the inequalities into two

$x - 5 \leq 2x + 1 < 10 - x$

If this is done correctly, your solution should be $-6 \leq x < 3$

Activity (*16.2*) Solving Inequalities

Questions

1) Solve the following simultaneous inequalities leaving your answer in the form a $< x \leq b$

   a) $-6 \leq 10x + 14 \leq 24$

   b) $-\frac{9}{4} < x + \frac{1}{4} < \frac{3}{8}$

   c) $-2x + 2 \leq 3x + 8 < 16$

2) Solve the following simultaneous inequalities leaving your answer in the form $x < a$ or $x \geq b$

   $7x - 3 \leq 4x < 7x - 12$

Answers

1 a) $\frac{4}{5} \leq x \leq 1$ b) $-2\frac{1}{2} < x < \frac{1}{8}$ c) $x \geq -1\frac{1}{5}$ or $x < 2\frac{2}{3}$

2) $x \leq 1$ or $x > 4$

16.3 GRAPHICAL REPRESENTATION OF INEQUALITIES

Inequality solutions can be represented graphically either

- On a number line or
- On the Cartesian plane

16.3.1 Representing inequalities on a number line

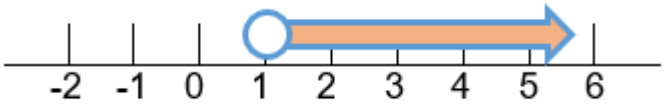

Fig 16.1 - Inequality on a number line

Tip; there are basic concepts for drawing and using the number line which are as follows;

1. The number is not included when we use < and >……do not shade the circle
2. The number is included when we use ≤ and ≥…….shade the circle

Let us consider a few examples on representation of inequalities on a number line.

Worked Example [10]

Questions

Represent the following on a number line and write the solution set represented by each number line

a) $x > 3$ b) $x \geq -1$ c) $x < 4$ d) $x \leq -2$

Solution

a) $x > 3$

the number line will have an unshaded circle above 3, meaning that 3 is excluded from the solution set

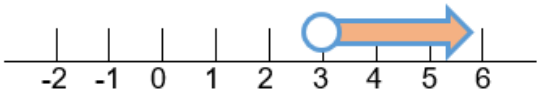

Fig 16.2

Solution set ; $x = \{4;\ 5;\ 6;\ \ldots\}$

b) $x \geq -1$

The number line will have a shaded circle above –1, meaning that –1 is included in the solution set

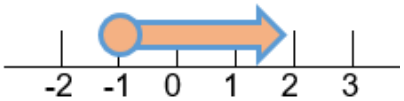

Fig 16.3

Solution set; $x = \{-1;\ 0;\ 1;\ \ldots\}$

c) $x < 4$

Do we shade or we do not shade the circle above 4 on the number line?.....

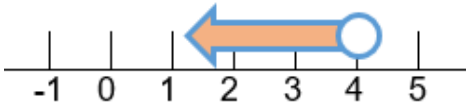

Fig 16.4

Solution set : $x = \{3;\ 2;\ 1;\ \ldots\ \}$

d) $x \leq -2$

Do this yourself, represent the inequality on the number line below

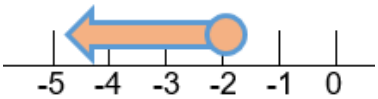

Fig 16.5

Solution set; $x = \{ \qquad \qquad \qquad \}$

Now let us consider a inequalities of the form $2 \geq x$ or $-3 < x$ and how we represent these on a number line.

Worked Example [11]

Questions

Represent the following on a number line and write down the solution set

a) $2 < x$

b) $-4 > x$

c) $-2\frac{1}{2} \geq x$

d) $4 \leq x$

Solution

a) $2 < x$

For you to easily draw this type of inequality you have to start by writing the variable first that is x in this case and it will be like $x > 2$ (you should not distort the meaning of the inequality)

Then we draw the number line as we have been doing in the previous examples.

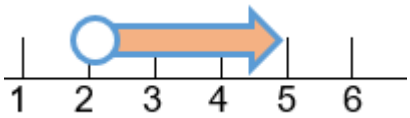

Fig 16.6

Solution set; $x = \{3;\ 4;\ 5;\ \ldots\}$

b) $-4 > x$

Start by writing x on the inequality.

$x < -4$ (you should not distort the meaning of the inequality)

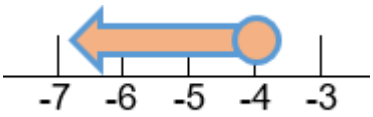

Fig 16.7

Solution set; $x = \{ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \}$ write the solution set yourself

c) $-2\frac{1}{2} \geq x$

Do this yourself

Write the inequality starting with x in the space below

Draw the number line here in the space below

Solution set $x = \{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\}$

Now let us represent simultaneous inequalities of the form $a < x \leq b$ on a number line

Worked Example [12]

Questions

Represent the following on a number line and give the solution set

a) $2 \leq x < 5$

b) $-2 < x < 1$

c) $-3 \leq x \leq 2\frac{1}{2}$

Solutions

a) $2 \leq x < 5$

You draw a number line like the one below.

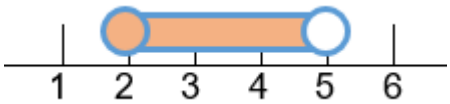

Fig 16.8

Solution set $x = \{2;\ 3;\ 4\}$

b) $-2 < x < 1$

Both circles are not shaded. Why is it so?

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Draw the number line in the space below.

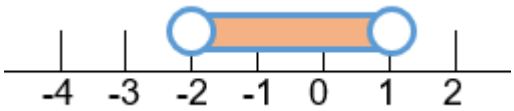

Fig 16.9

Solution set $x = \{-1;\ 0\}$

c) $-3 \leq x \leq 2\frac{1}{2}$

Both circles are shaded. Why is that so?

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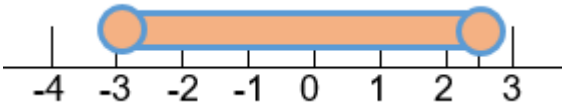


Fig 16.10

Solution set $x = [-3;\ -2;\ -1;\ 0;\ 1;\ 2, 2\frac{1}{2}]$

Having done this lets represent simultaneous inequalities of the form $x \leq a$ or $x > b$ on the number line

Worked Example [13]

Questions

Represent the following on a number line and give the solution set

a) $x \leq -2, x > 1$
b) $x < 3, x > 6$
c) $x < -6, x \geq -2$

Solutions

a) $x \leq -2, x > 1$

These two inequalities are drawn on the same number line

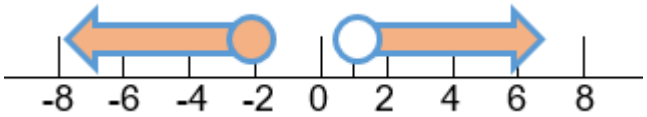


Fig 16.11

Solution set; $x = \{-2; -3; -4; \ldots\}$ or $x = \{6; 7; 8; \ldots\}$

b) $x < 3$ or $x \geq 6$

We draw the inequalities on the same diagram

Draw the number lines in the space provided

Solution set; $x = \{2; 1; 0; -1; \ldots\}$ or $x = \{6; 7; 8; \ldots\}$

Activity (*16.3*) Inequalities on a number line

Questions

Represent the following on a number line

a) $x < -1$

b) $x > 4$

c) $-3 < x \leq 2$

d) $-7 \leq x < -1$

16.3.2 Illustrating an inequality on the Cartesian plane

What is a Cartesian plane? Yes it is the y and the x–axis as shown below

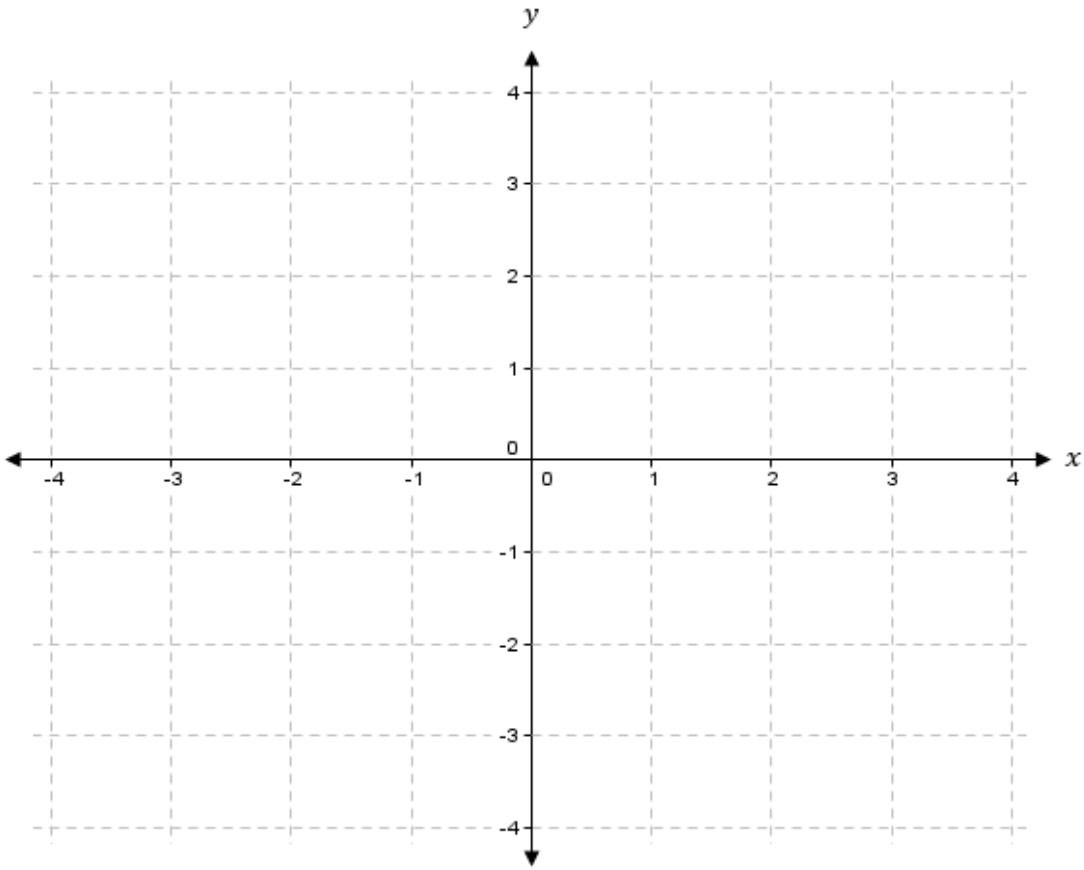


Fig 16.13

You are going to plot horizontal lines, that is inequalities of the form $y \leq 2$and/ or $y > -1$ and shade the unwanted regions

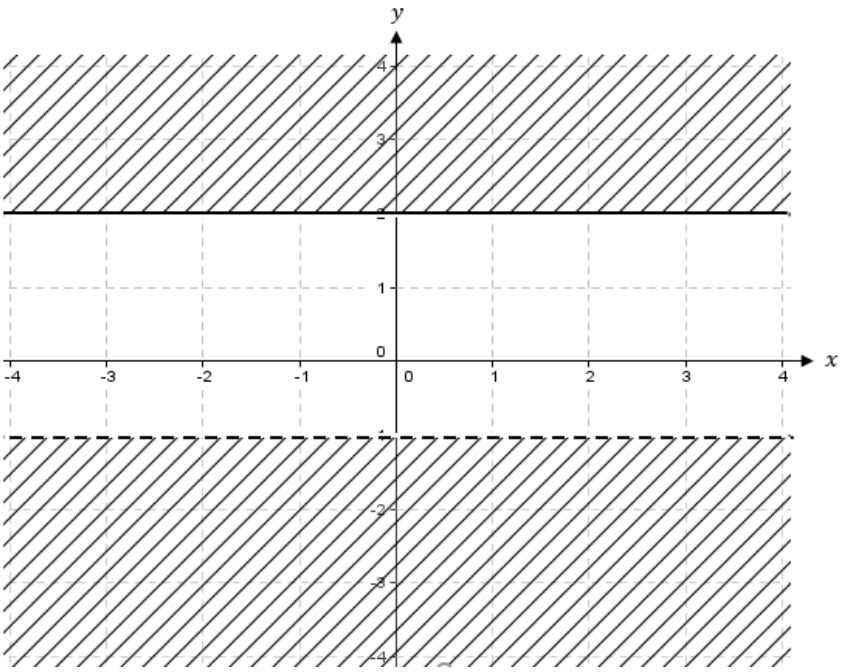


Fig 16.14

You are also going to plot vertical lines, that is inequalities of the form $x \geq 2$ and / or $x < -3$ and shade the unwanted region

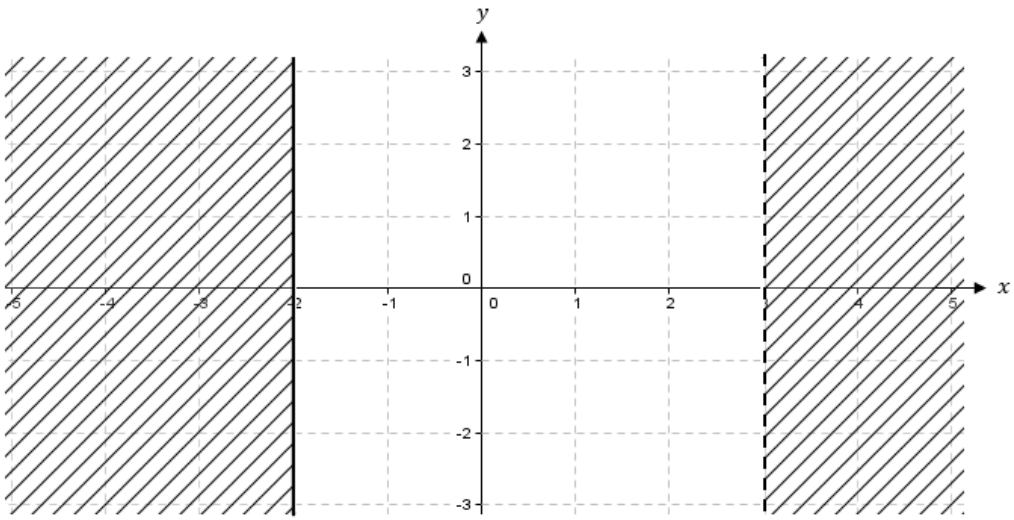


Fig 16.15

And you are also going to draw slant line or oblique lines, that is inequalities of the form $y \leq mx + c$ and shade the unwanted regions

Fig 16.6 shows the graph of $y \leq 2x + 4$

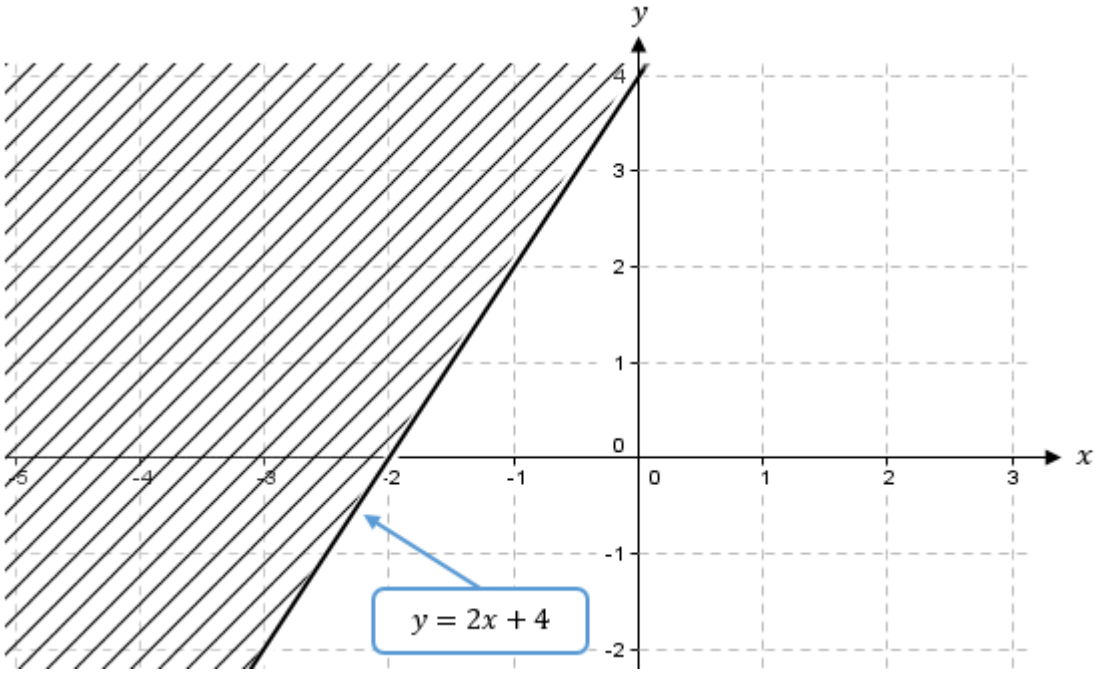


Fig 16.16

Let us consider an example in which you learn how to draw the lines and shade the unwanted region

Worked Example [14]

Questions

Represent the following on a Cartesian plane

a) $x > 2$

b) $x \leq -1$

c) $-2 < x \leq 3$

d) $1 \leq x \leq 5$

Solutions

a) $x > 2$

Tip in the Cartesian plane we do not write inequality signs we use equal signs. We draw a line passing through $x = 2$ and label it, the line is also dotted since we are using >.

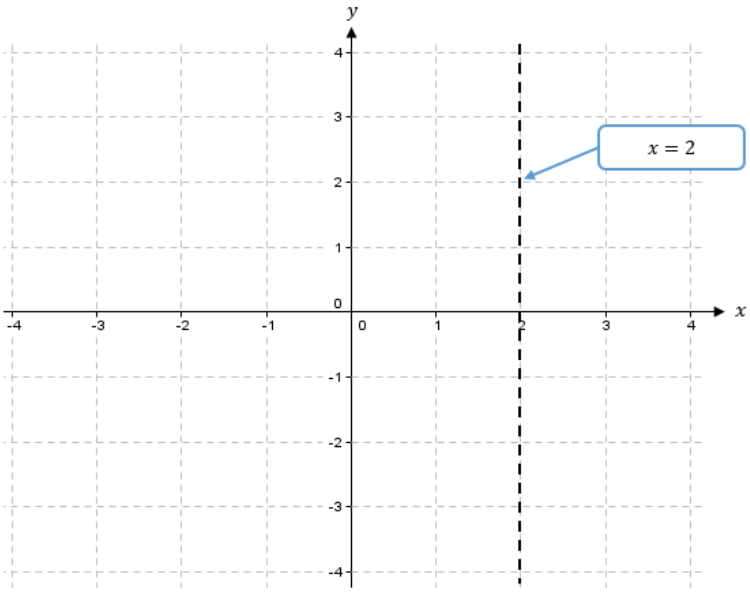


Fig 16.17

We also shade the values which are not greater than 2 since x is greater than 2

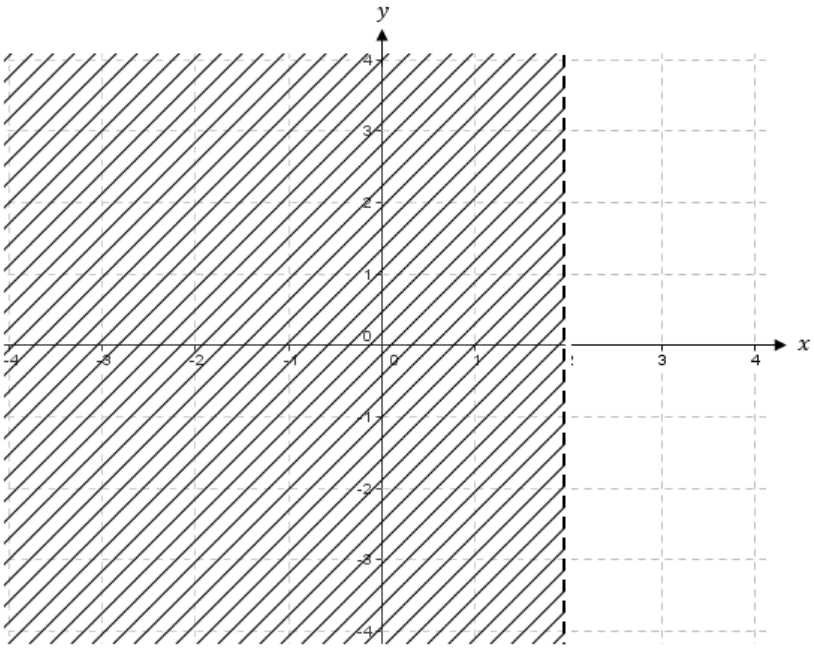


Fig 16.18

a) $x \leq -1$

Tip; since you are using ≤, we are also using a bold line

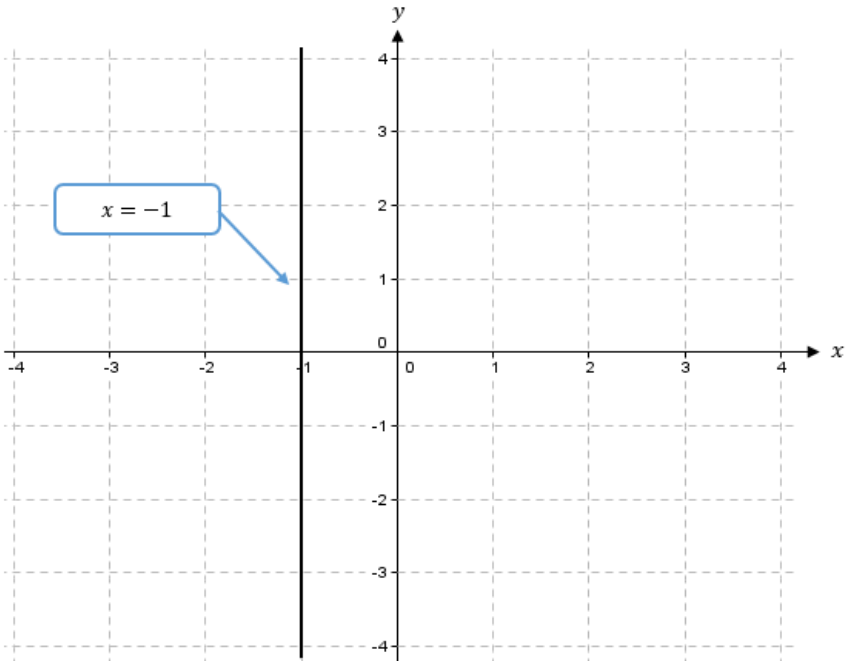


Fig 16.19

Which side do you shade? Which side is the unwanted region. Using the diagram above shade the unwanted region 'below is how it should be after you have shaded the unwanted region

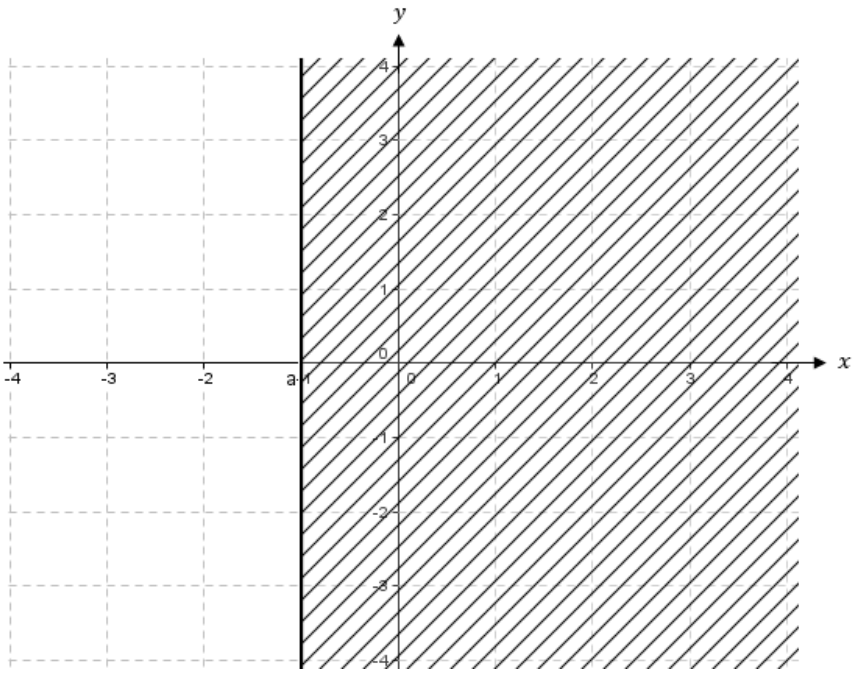


Fig 16.20

b) $-2 < x \leq 3$

Here x is any value or all values between –2 and 3. Therefore anything which is not between -2 and 3 is the unwanted region.

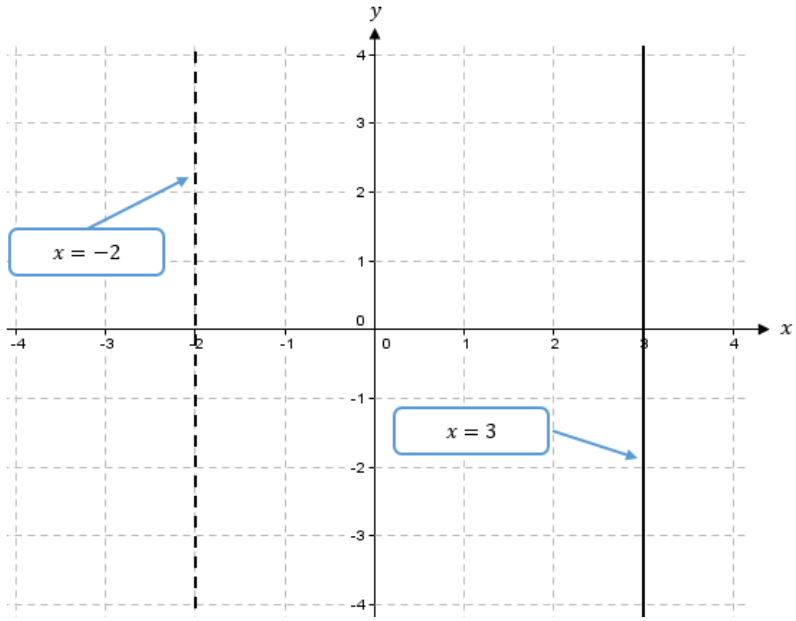


Fig 16.21

Now shade the unwanted region on the diagram above and it should be like the one below

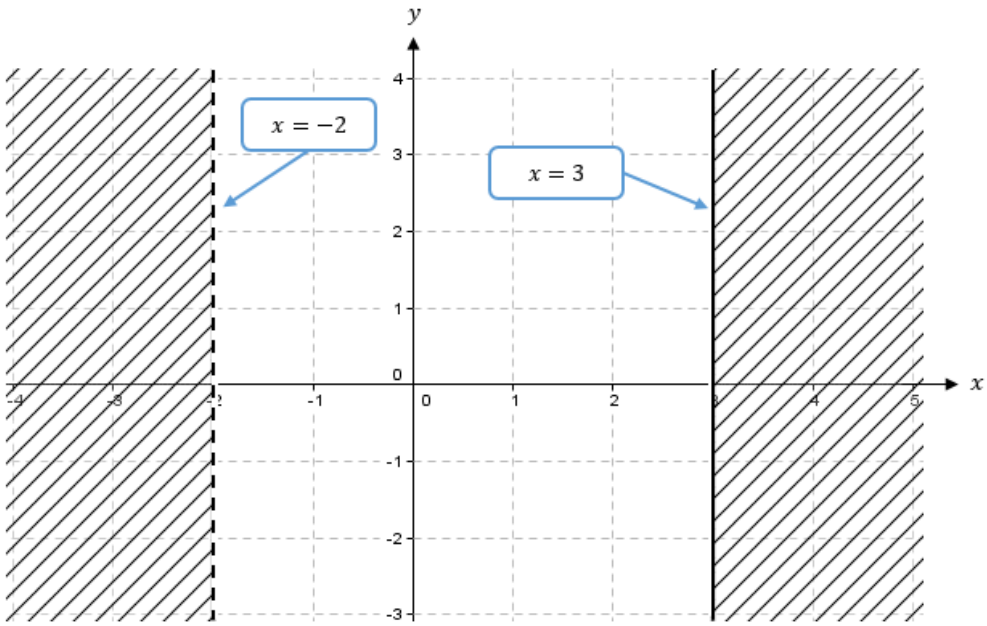


Fig 16.22

c) $1 \leq x \leq 5$

Remember the values of x are between 1 and 5

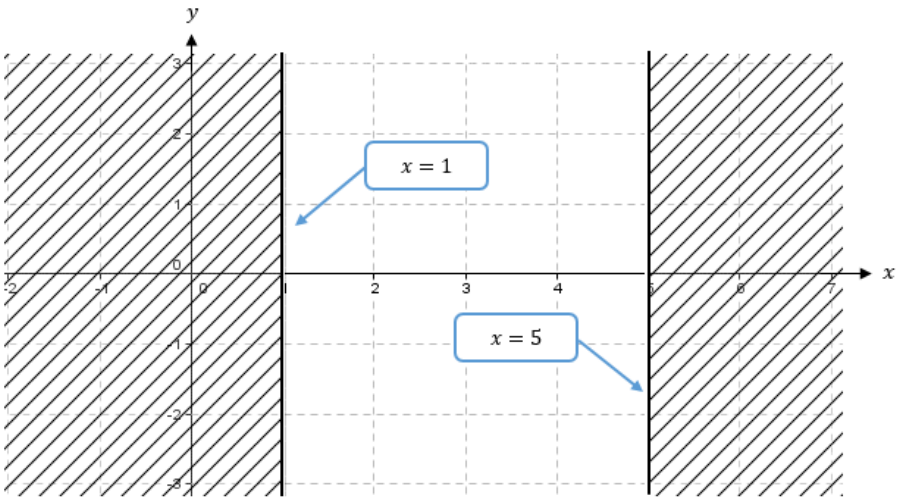


Fig 16.23

Let us consider the following example in which you will learn to draw horizontal inequality lines and shade the unwanted region.

Worked Example [15]

Questions

Represent the following on the Cartesian plane

a) $y > 2$
b) $y \leq -2$
c) $-1 < y \leq 3$
d) $1 \leq y \leq 5$

Solutions

a) $y > 2$

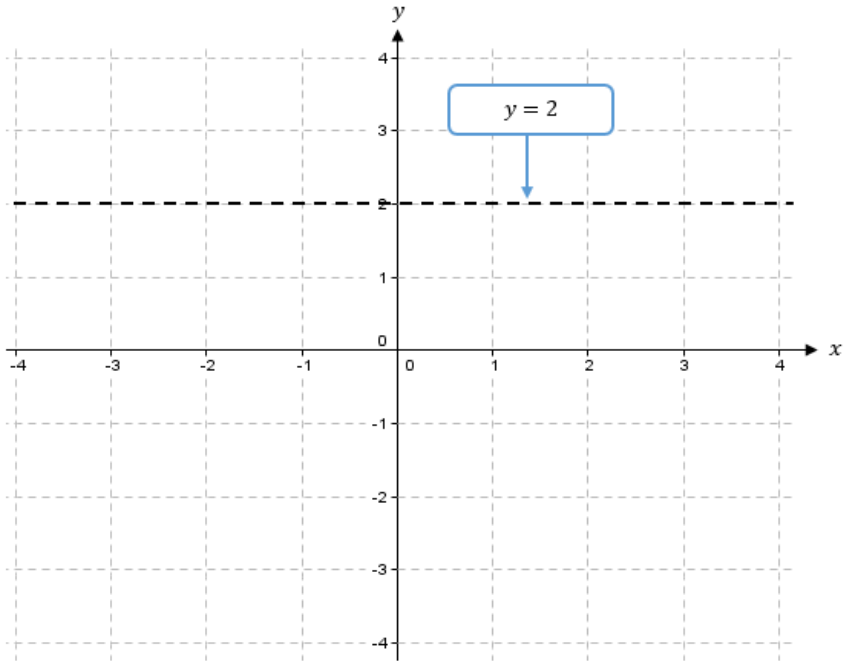


Fig 16.24

Use the diagram above to shade the unwanted region. The solution should look like the one below.

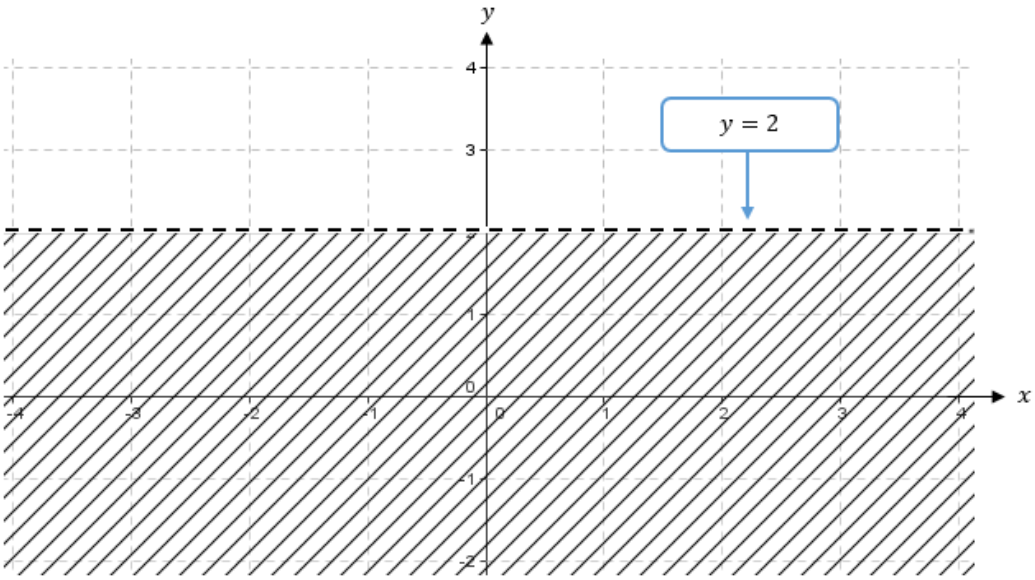


Fig 16.25

b) $y \leq -2$

the y values here are the values less than or equal to –2 therefore the y values are –2,; –3; –4 ; –5; . . . so any other numbers lies in the unwanted region. In the diagram below shade the unwanted region

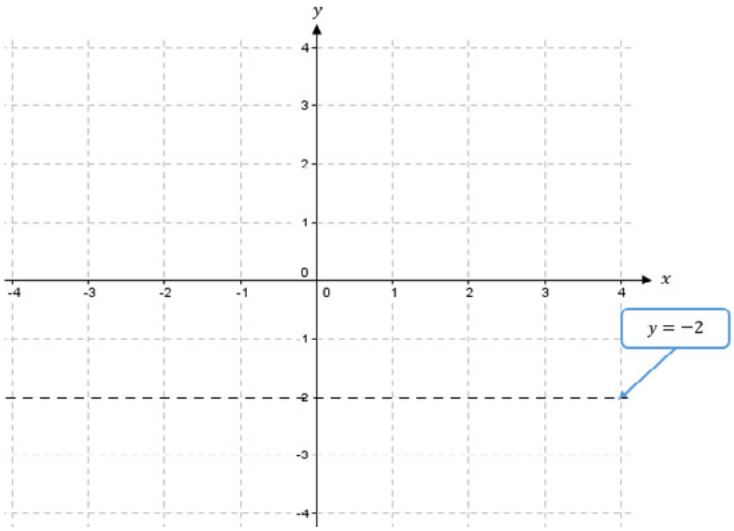


Fig 16.26

c) $-2 < y \leq 3$

Here the y values are the values between –2 and 3 therefore any other numbers lie in the unwanted region. Remember to use bold lines and dotted where they apply.

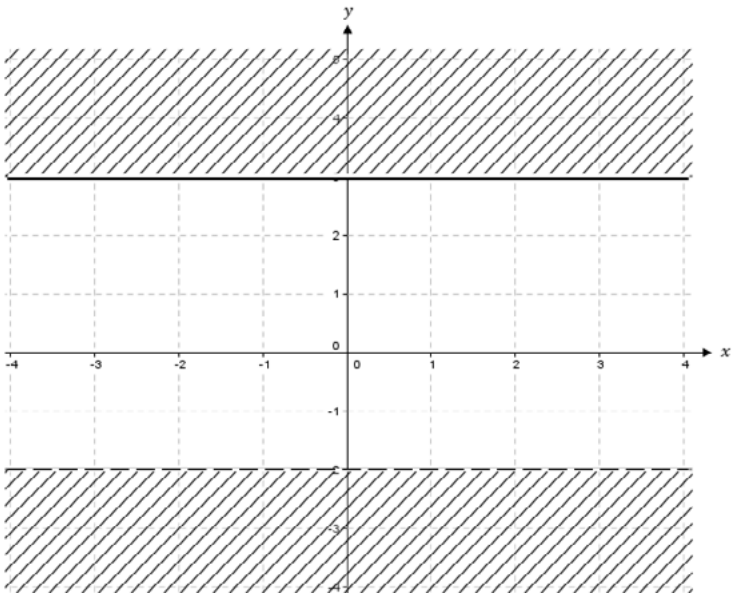


Fig 16.27

d) $2 \leq y \leq 5$

Here the y values are the values between 2 and 5 therefore any other values lie in the unwanted region and you shade the unwanted region. In the diagram below shade the unwanted region.

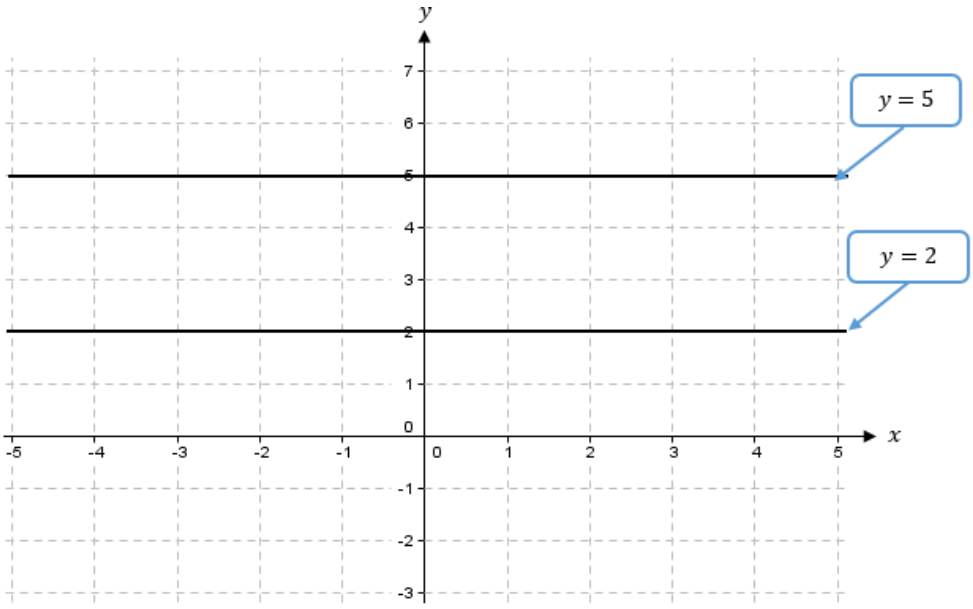


Fig 16.28

Oblique lines / slant lines on the Cartesian plane

Slant inequalities will be in the form $y \leq mx + c$. How then do we represent such inequalities on the Cartesian plane? How then do we shade the unwanted region. In this section you will learn how to represent slant lines in the Cartesian plane and how to shade the unwanted region.

Step 1 – We convert the inequality to an equation, that is from $y \leq mc + c$ to the form $y = mx + c$

Step 2 – Then we formulate a table of values for the equation

Step 3 – Use the table of values to plot the line on the Cartesian plane

Tip; < and > we use a broken like

≥ and ≤ we use bold line

How do we shade the unwanted region?

Step 4 – to shade the unwanted region one has to observe that the line on the Cartesian plane have divided the Cartesian plane into two sections.

You then have to choose any coordinates from any of the two sides preferably the coordinates to choose is the origin that is (0; 0) and input these values in the inequality in question.

After inputting and simplifying, the result can either be a true mathematical statement say 0 < 4 then it means that the coordinates are from the wanted region then you just have to shade the other side.
Say you have incorporated the coordinates in the inequality and the result after simplifying is a false mathematical statement say 3 > 5 then it means the selected coordinates are from the unwanted region and you shade that side.

Let us consider an example on how to draw slant inequalities and how to decide on the unwanted region.

Worked Example [16]

Questions

Represent the following on a Cartesian plane

a) $y \leq 2x + 4$ b) $2y > -x + 2$

c) $2x + 3y < 12$; $y \leq x + 4$

Solutions

a) $y \leq 2x + 4$

Change the inequality to an equation, it becomes $y = 2x + 4$ and then make a table of values

x	0	
y		0

When $x = 0$ (the value of y is obtained by substituting 0 in the equation)

$y = 2x + 4$

$y = 2(0) + 4$

$y = 0 + 4$

$y = 4$

When $y = 0$ (the value of x is obtained by substituting in the equation)

$y = 2x + 4$
$0 = 2x + 4$
$-4 = 2x$
$-2 = x$ (after dividing by 2)

x	0	–2
y	4	0

We take the coordinates from the table of values and mark them on the Cartesian plane

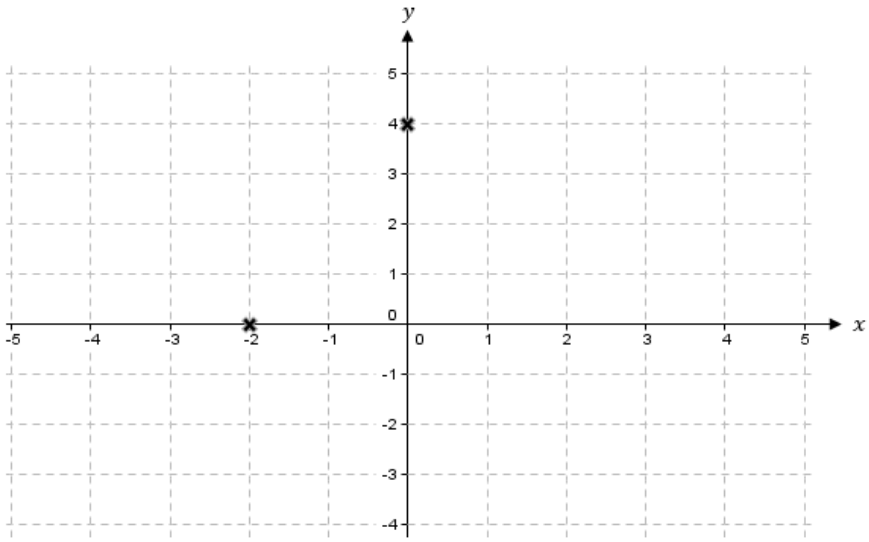


Fig 16.29

We then join the marked coordinates with a straight line but remember to use a bold line since we are using ≤.

Draw the line on the above diagram, joining the coordinates with a straight line. The line would look like the one below

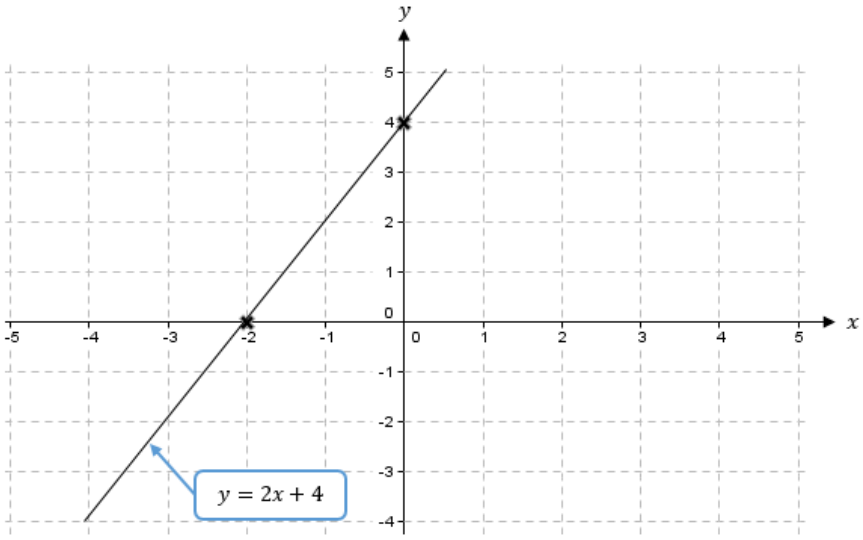


Fig 16.30

Now which side are we going to shade? Which side is the unwanted region? We can see that the line on the Cartesian plane has divided the Cartesian plane into 2 sections.

Let us take coordinates from any one of the sides and input in the inequality and find out if it yields a true statement (wanted region) or false statement (unwanted region, and we shade that side whose coordinates have obtained a false mathematical statement).

Taking coordinates (0; 0)

$y \leq 2x + 4$

$0 \leq 2(0) + 4$

$0 \leq 0 + 4$

$0 \leq 4$ (looking at this statement it is mathematically true on the ground that 0 is truly less than 4, ignore the or equal part)

Since we have accepted the statement as true then we shade the other side of the line as the unwanted region and consider the side with the points (0;0) as the wanted region.

Using this, on the above diagram shade the unwanted region

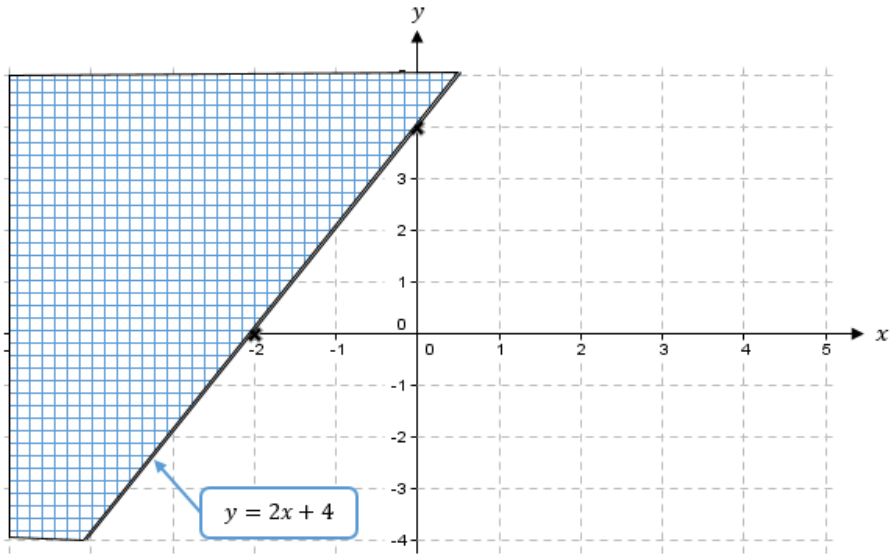


Fig 16.31

b) $2y > -x + 2$

Do this yourself; First change the inequality to an equation

Write the equation in the space below

Then we make the table of values for $2y = -x + 2$

x	0	
y		0

When $x = 0$ then we obtain the value of y by substituting 0 in the equation. Use the space below

When $y = 0$ you have to substitute again in the equation to find the value of x.

Use the space below to find the x value

If all this have been done correctly the table of values will be like the one below

x	0	2
y	1	0

We then take the coordinates from the table of values and mark the points on the graph.

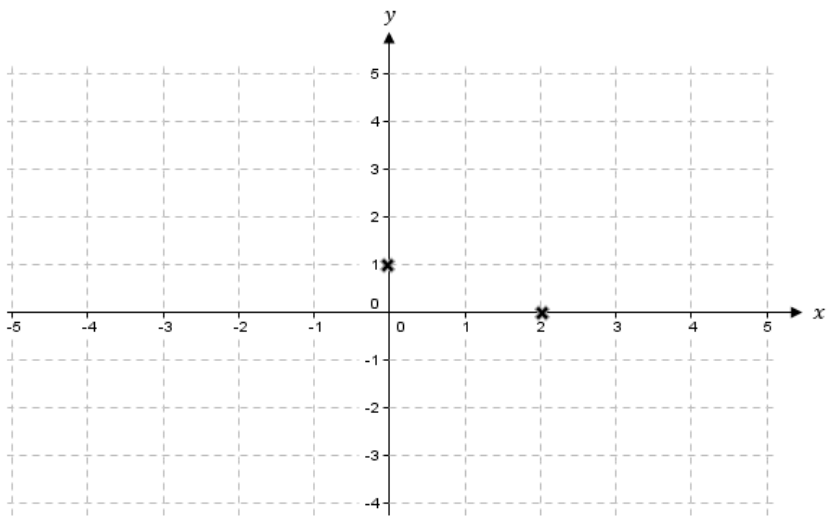


Fig 16.32

Remember, before plotting the line, it should be dotted because we are using >. On the above diagram draw the line and shade the unwanted region, for any workings use the space below

If all is done correctly the graph should look like the one below.

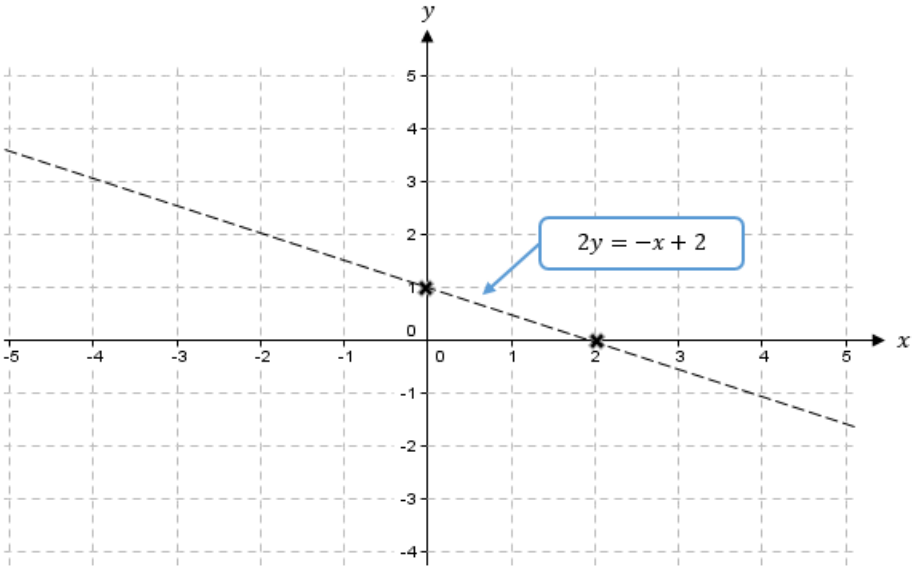


Fig 16.33

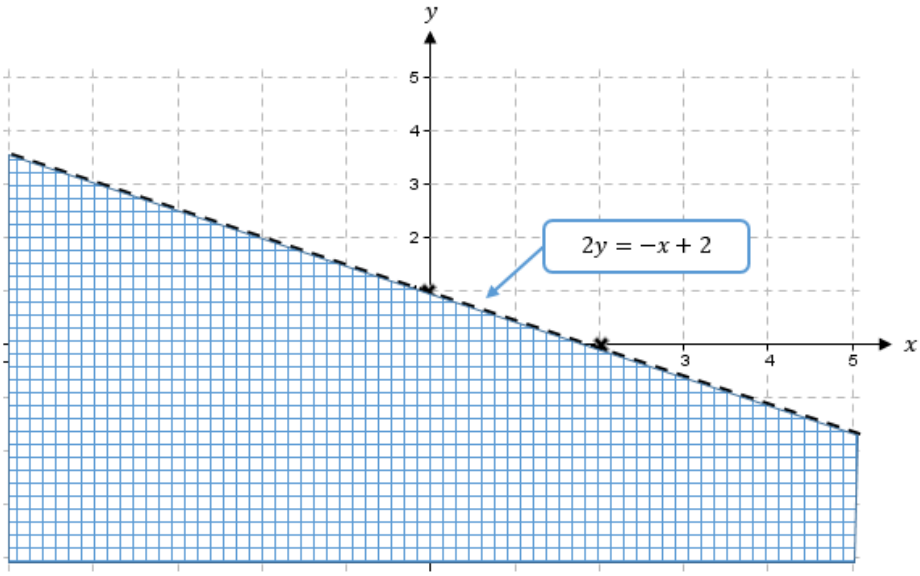


Fig 16.34

c) $2x + 3y < 12$; $y \leq x + 4$

This requires you to draw the two inequality lines on the same Cartesian plane.

Do this yourself

Convert all the inequalities to equations

$2x + 3y < 12$ becomes .....

$y \leq x + 4$ becomes .....

Now you make a table of values for each equation

$2x + 3y = 12$	$y = x + 4$
<table><tr><td>x</td><td>0</td><td></td></tr><tr><td>y</td><td></td><td>0</td></tr></table>	<table><tr><td>x</td><td>0</td><td></td></tr><tr><td>y</td><td></td><td>0</td></tr></table>
Work out the missing values here	Work out the missing values

Use the coordinates from the table of values and plot the two lines

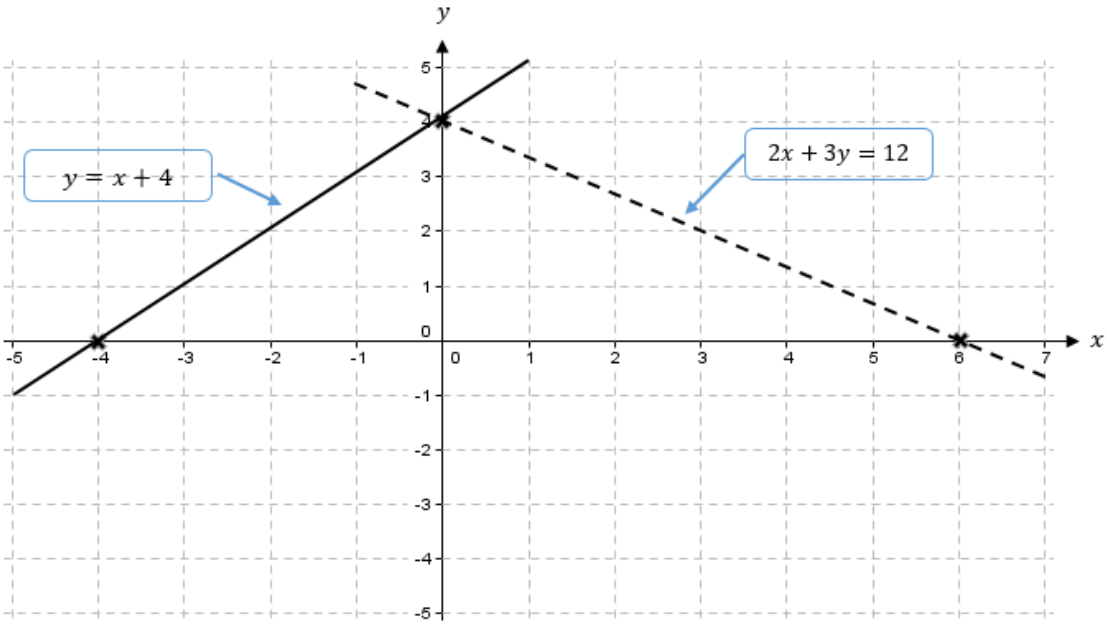


Fig 16.35

Now you have to decide on the unwanted region. Shade the unwanted region on the above graph and any working to be done should be done in the space below

If all working have been done correctly you should have a diagram like the one below

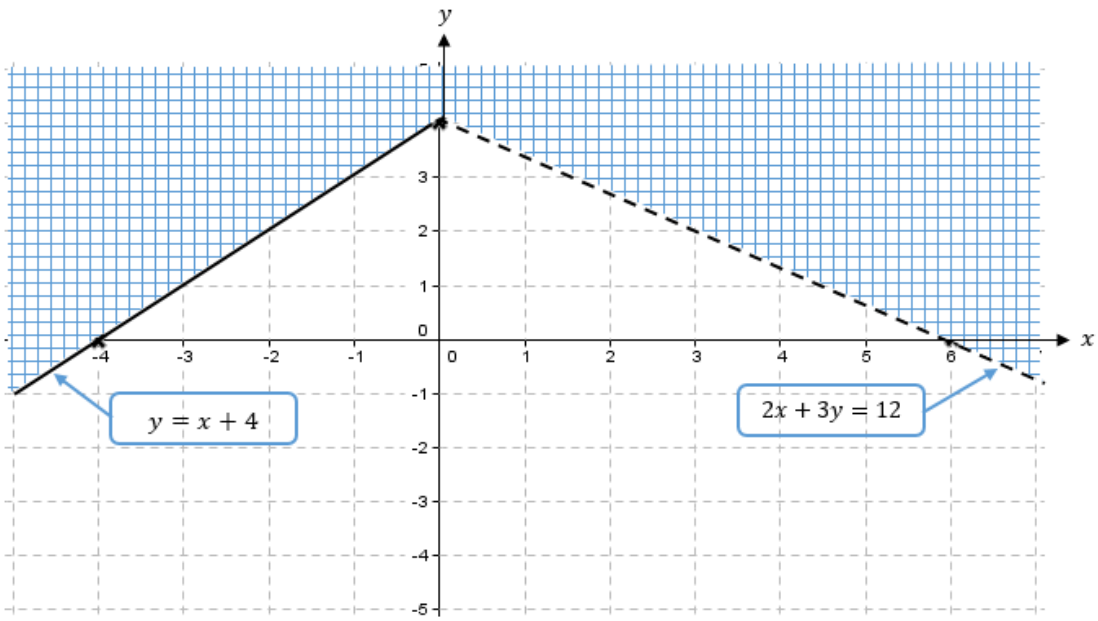


Fig 16.36

Activity (*16.4*) Inequalities on a Cartesian plane

Questions

Represent the following on a Cartesian plane

1. $3y \geq 2x + 3$
2. $y > 2x + 1$
3. $y \leq x$
4. $3x - y < 9$
5. $3y - 2x \geq 6$; $2y < -x + 2$

16.4 LINEAR PROGRAMMING

Where do you think inequalities can be applied in real life situations? Inequalities can be applied in the business world in finding maximum cost, maximum profit or even the best method of production. In linear programming we use the Cartesian plane to obtain solutions, especially given restrictions or constraints.

What is Linear Programming?

This is a method in which all the requirements are met in a given situation. It involves converting real life situations into inequalities and present them on a Cartesian plane. In this section we are going to look at converting practical problems into inequalities, translating each verbal constraint into a linear inequality, plotting the inequality and shading the unwanted region.

Procedure followed when solving Linear programming problems

Step 1 – Translate verbal constraints into linear inequalities

Step 2 – Plot the linear inequalities on a Cartesian to obtain the accessible region (the wanted region) which satisfies all the inequalities (by shading the unwanted region). The accessible region is that region which is not shaded.

Tip; At least means $\geq$

At most means $\leq$

Let us consider an example in which we will learn to convert verbal constraints to linear inequalities and represent them on the Cartesian plane.

Worked Example [17]

Questions

A builder wishes to build houses and flats on 6000 m² plot of land

a) The city council insists that there must be more than 6 houses and that there must be more flats than houses .

Taking x to represent the number of houses and y to represent the number of flats, write down two inequalities other than $x > 0$ and $y > 0$ which satisfy these conditions

b) The builder allows 400 m² for each house and 300 m² for each flat. Write down another inequality which satisfy this condition and show that it reduces to $4x + 3y \leq 60$

c) The point (x; y) represents x house and y flats. Using a scale of 2 cm to represent 5 units on both axes, draw the x and y axes for $0 \leq x \leq 20$ and $0 \leq y \leq 20$. Construct and show by shading the unwanted region, the region in which (x; y) must lie.

Solutions

Tip: The first statement gives us the constraint that is in whatever the builder builds, the limit is 6000 m ² of land. The number of m² for house plus the number of m² for flats should be less than 6000 m².

a) Now let us translate the verbal statements to inequalities

Houses : x

Flats : y

There must be more than 6 houses will translate to $x > 6$

There must be more flats than houses will translate to $y > x$

b) Builder allows 400 m² for each house will translate to $400x$

Builder allows 300 m² for each flat will translate to $300y$

Since the limit of land or the land available here is 6000 m², the sum of the two should not exceed 6000 m²

This will translate to

$400x + 300y \leq 6000$ (we divide by 100 both sides to reduce it and as well show as per the requirement of the question.

We will reduce to $4x + 3y \leq 60$

c) Using the given scale you draw the Cartesian plane and convert all inequalities to equations

$x > 6$ will be $x = 6$

$y > x$ will be $y = x$

$4x + 3y \leq 60$ will be $4x + 3y = 60$

Make table of values for the two equations

<table>
<tr><td colspan="2">$y = x$</td><td colspan="2">$4x + 3y = 60$</td></tr>
<tr><td colspan="2"><table><tr><td>x</td><td>0</td><td>10</td></tr><tr><td>y</td><td></td><td></td></tr></table></td><td colspan="2"><table><tr><td>x</td><td>0</td><td></td></tr><tr><td>y</td><td></td><td>0</td></tr></table></td></tr>
<tr><td colspan="2">When $x = 0$
$y = 0$

When $x = 10$
$y = 10$</td><td colspan="2">When $x = 0$
$4(0) + 3y = 60$
$3y = 60$ (divide by 3)
$y = 20$

when $y = 0$
$4x + 3(0) = 60$
$4x = 60$ (divide both sides by 4)
$x = 15$</td></tr>
<tr><td colspan="2"><table><tr><td>x</td><td>0</td><td>10</td></tr><tr><td>y</td><td>0</td><td>10</td></tr></table></td><td colspan="2"><table><tr><td>x</td><td>0</td><td>15</td></tr><tr><td>y</td><td>20</td><td>0</td></tr></table></td></tr>
</table>

Using the coordinates from the table of values we plot lines on the Cartesian plane and shade the unwanted region as shown below

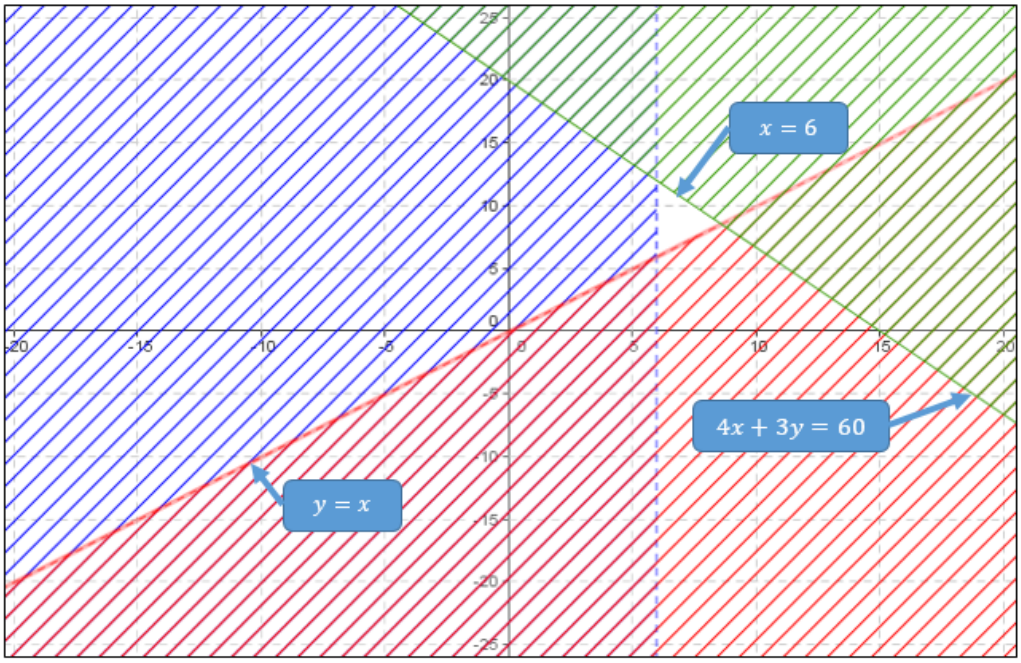


Fig 16.37

Let us consider another example.

Worked Example [18]

Questions

Mr Phiri visits Gorilla farmers store to buy fertilizer and seed packs. He decides to buy at least 4 bags of fertilizer and at least 2 packs of seed. If x represents the number of fertilizer bags and y represents seed packs bought,

1) (a) Write down two inequalities which satisfy the above conditions
   (b) Mr Phiri has only $ 1200 to spend. A bag of fertilizer cost $150 each and a pack of seed cost $60 each. Write down another inequality and show that it reduces to $5x + 2y \leq 40$.
   (c) Mr Phiri intends to buy less seed packs than fertilizer bags. Write down another inequality which satisfies the condition

2) The point $(x;\ y)$ represents x bags of fertilizer and y packs of seed. Using a scale of 2 cm to represent 1 bag of fertilizer on the horizontal axis and 2 cm to represent 2 seed packs on the vertical axes, draw and indicate by shading the unwanted region the region in which $(x;\ y)$ must lie.

Solutions

1) Note that
   - x represents fertilizer and
   - y represents seed packs

a) The statement 'at least 4 bags of fertilizer' can be represented as $x \geq 4$
The statement 'at least 2 packs of seed' can be represented as $y \geq 2$

The limit is \$1 200 for everything bought, that is, the sum of things bought should not exceed \$ 1200.

- The number of fertilizer bought (x) times the unit price of each fertilizer bag (\$150) gives $150x$.
- The number of seed packs bought (y) times the unit price of each seed pack (\$60) gives $60y$.

b) The inequality produced is

$$150x + 60y \leq 1200$$

We now divide all terms by 30 to reduce everything to lowest terms

$\frac{150x}{30} + \frac{60y}{30} \leq \frac{1200}{30}$ to obtain $5x + 2y \leq 40$

c) The inequality for the statement 'less seed packs than fertilizer bags' is $y < x$

2) Now we have four inequalities which need to be represented on the Cartesian plane using the given scale. You first convert the inequalities to equations

$x \geq 4$ becomes $x = 4$

$y \geq 2$ becomes $y = 2$

$5x + 2y \leq 40$ becomes $5x + 2y = 40$

$y < x$ becomes $y = x$

Now you make a table of values for equations with slant lines

$5x + 2y = 40$

x	0	
y		0

When $x = 0$, find the value of y by substituting in the equation above.

When $y = 0$, find the value of x by substituting in the equation above.

If all is done correctly the table of values should be like the one below

x	0	8
y	20	0

$y = x$

x	0	10
y		

When $x = 0$, find the value of y by substituting in the equation above.

When $x = 10$, you need again to find the value of y by substituting in the equation above

If all is done correctly the table of values should be like the one below

x	0	10
y	0	10

Now using the coordinates from the table of values, we plot the lines on the Cartesian plane

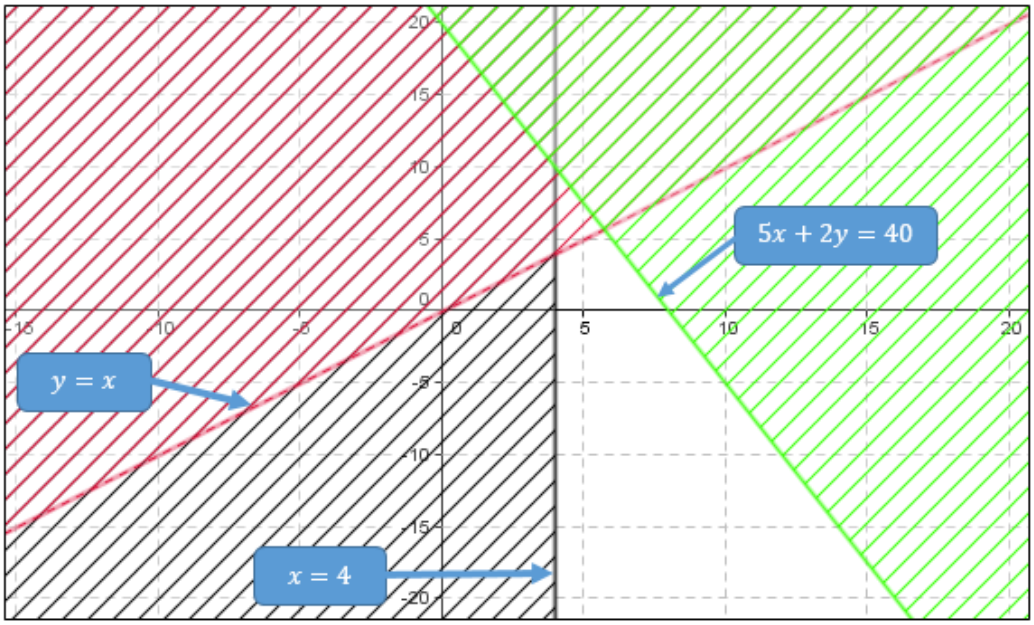


Fig 16.38

Finding the maximum or minimum values in Linear programming problems

What do you understand by the terms maximum and minimum? Maximum means the highest or greatest and minimum means lowest or least/smallest.

Once the region of accessible points has been established, the next stage is to maximize or minimize. It is worth noting that the maximum or minimum occur mostly at the point of intersection of two boundaries.

Remember not to include those numbers on the dotted line and also remember that we are using integer values only.

Let us consider an example and find out how we calculate the maximum or minimum.

Worked Example [19]

Questions

Using the diagram or the solution of Example 17,

Use the graph to find

a. The maximum number of flats that can be built
b. The maximum number of houses that can be built
c. The values of x and y which gives the maximum number of dwelling units

Solutions

a) Remember (y) represents flats.

Tip: do not take values on the dotted line. Use the diagram below.

The highest value of y in the accessible region is y=8

b) Remember (x) represents houses. which value in the accessible region is the greatest x value

The highest value of x in the accessible region is x=12

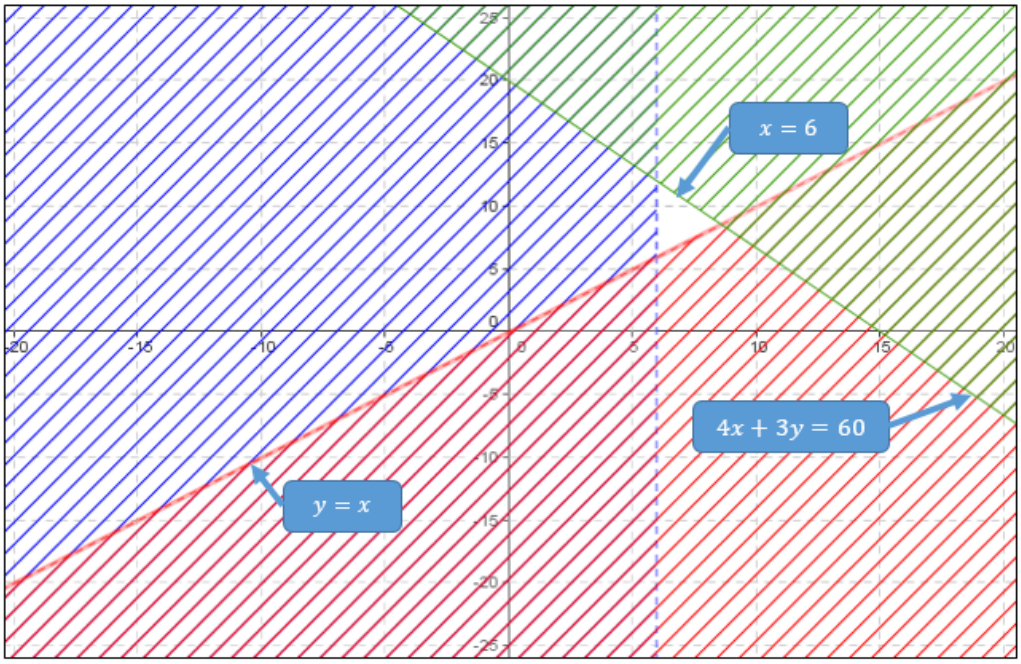


Fig 16.39

c) The question requires you to find the combination of coordinates in the accessible region which gives the highest number of dwelling units. Here you need to look for coordinate positions where if x and y are added they produce the highest or maximum values.

Tip: we usually use trial and error to find the combination we need. You test those numbers in the extremes

Therefore from the diagram, coordinate (6; 12) gives the highest number of dwelling units.

Activity (*16.5*) Linear programming

Questions

From the diagram on Worked Example 18, find

1) The number of possible combinations of bags of fertilizer and seed packs Mr. Phiri could buy.
2) The combination which is equals all the money available to Mr. Phiri.

Answers

Do this yourself

Mark all the possible combinations in the unshaded region

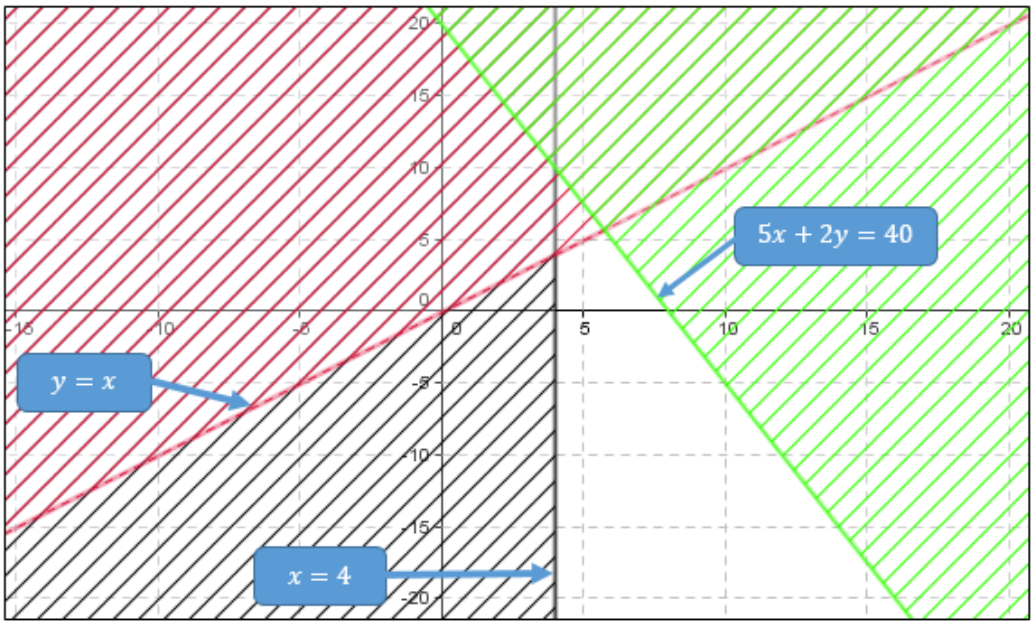


Fig 16.40

16.5 Summary

This unit covered Inequalities where it looked at how to solve inequalities, representation of inequalities graphically and linear programming. If you had understood concepts on algebra then you didn't have problems with this unit.

16.6 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

16.7 Assessment Test

1) Mrs. Phiri bakes bread and buns using flour and wheat. A dozen loaves of bread requires 7 kg of flour and 4 kg of wheat. A dozen buns requires 3kg of flour and 2 kg of wheat. Mrs. Phiri had 55 kg flour and 49 kg wheat. Let x be the dozens of bread and y the dozen of buns.
   a) Using the information, write down two inequalities in x and y which satisfy these conditions
   b) In order for Mrs. Phiri to make profit, she should manufacture more than 5 dozens of bread and at least 7 dozens of buns. Write down two inequalities which satisfy these conditions
   c) The point (x; y) represent x dozen of bread and y dozens of buns baked. Indicate clearly by shading the unwanted regions, the region in which (x; y) must lie.
   d) Using your graph, write down all possible combinations which give the maximum number of dozens of bread and buns manufactured.

2) Mrs. Banda visited Mbare Musika to buy some tomatoes and onions for her wedding celebrations. She decide to buy at least 9 boxes of tomatoes and at least 5 boxes of onions. If x is the number of boxes of tomatoes and y is the number of onions.
   a) (i) Write the inequalities which satisfy the above conditions.

(ii) Mrs. Banda has \$120 to spend. A box of tomatoes costs \$6 and a box of onions costs \$8. Write down an inequality in x and y and show that the inequality will reduce to $3x + 4y \leq 6$

(iii) Mrs. Banda intended to buy less boxes of onions than of tomatoes. Write down another inequality in x and y

b) Show by shading the unwanted regions, the region in which x and y must lie.

c) From the graph, find;

(i). The number of possible combinations of boxes of tomatoes and onions.

(ii). The combinations which equals all the money Mrs. Banda has.

UNIT 17 - TRIGONOMETRY RATIOS 1

CONTENTS

17.1 INTRODUCTION

Pythagoras, a Greek Mathematician discovered Pythagoras' theorem. It is because of his passion in Mathematics, and Geometry in particular that we widely use his mathematical discovery. In this unit we are going to learn about Pythagoras' theorem and its application. We are also going to learn about Trigonometric ratios.

OBJECTIVES

After going through this unit, you should be able to;

- Use Pythagoras' theorem to solve problems involving right-angled triangles.
- Use scientific calculator to find the square and square roots of numbers.
- Define and use the tangent, sine or cosine in solving any acute right-angled triangles.
- Apply Pythagoras' theorem to solve real-life problems.

KEY WORDS

You are going to meet the following words and you are expected to know them

Acute angle - is an angle which is between 0 and 90 degrees.

Cosine angle - is a ratio that is defined by adjacent side over the hypotenuse side of a right angled triangle.

Angle of depression – is an angle of lowering to the ground level.

Angle of elevation – is an angle of rising from a level ground.

Hypotenuse side – is the longest side of a right-angled triangle and is opposite the 90 degree angle.

Obtuse angle – is an angle greater than 90 but less than 180 degrees.

Pythagoras – was a famous Greek Mathematician (philosopher) who discovered the rule or formula we call Pythagoras Theorem.
Sine angle – is a ratio that is defined by the opposite side over the hypotenuse side of a right-angled triangle.

TIME

You are advised not to spend more than 10 hours on this unit.

STUDY SKILLS

Most mathematical concepts are usually mastered after going through some sort of practice. You are therefore advised to solve as many questions on Trigonometric ratios as possible for you to grasp some of the concepts in this topic. Make sure you set your scientific calculator to indicate degrees by pressing DRG button on the calculator before you attempt any calculations on this unit.

17.2 PYTHAGORAS' THEOREM

Pythagoras' theorem states the relationship between the lengths of the three sides of a right-angled triangle. The theorem is used to calculate the length of a missing side in a right-angled triangle. A right-angled triangle has one of its angles equal to 90°, in this module this sign means 90° or right angle.

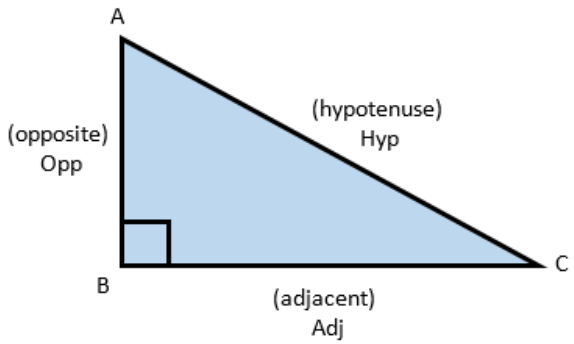


Fig 17.1 – Right angled triangle

Using Pythagoras' rule in the diagram above, the length of the hypotenuse AC is found as:

$$AC^2 = AB^2 + BC^2$$
$$AC = \sqrt{AB^2 + BC^2}$$

Now let us learn more about right-angled triangles:

- The side opposite to the right angle is the longest and is called hypotenuse.
- One acute angle used as reference angle **a** will give us opposite side.
- The side which forms or define our reference angle is called adjacent side.

Let us look at some examples on how to use Pythagoras's theorem to calculate the lengths of the sides of a right-angled triangle.

Worked Example [1]

Questions

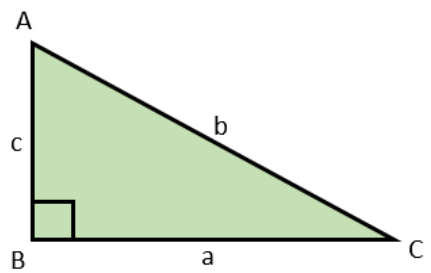


Fig 17.2

a. Calculate length b.
b. Calculate length a.

Solutions

a) $a^2 + c^2 = b^2$
b) $b^2 - c^2 = a^2$

Worked Example [2]

Questions

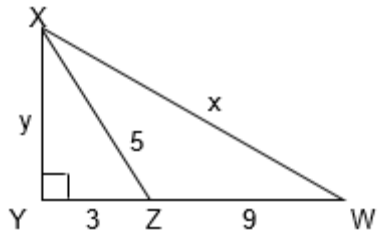


Fig 17.3

a) Calculate length y.

b) Calculate length x.

Solutions

a) $y^2 = 5^2 - 3^2$

$= 25 - 9$

$= \sqrt{16}$

$y = 4$

b) y2 + 122 = x2

$x^2 = 4^2 + 12^2$

$= 16 + 144$

$= \sqrt{160}$

$x = 12.6$

Worked Example [3]

Questions

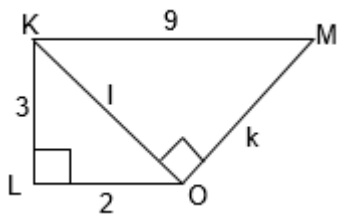


Fig 17.4

a) Calculate length l.

b) Calculate length k.

Solutions

a) $l^2 = 3^2 + 2^2$

$= 9 + 4$

$= \sqrt{13}$

$= 3.606$

$l = 3.61$

b) $k^2 = 9^2 - 3.61^2$

$= 81 - 13$

$= \sqrt{68}$

$k = 8.25$

Now that we have gone through some examples, let us see how you do on the following activity.

Activity (*17.1*) Pythagoras' theorem

Questions

XYZ is a right-angled triangle, angle Y = 90°. Find the length of the missing side. (hint you are advised to sketch and label your diagrams)

1. XY = 3cm, ZY = 4cm
2. XY = 12m, ZY = 9m
3. XZ = 13cm, ZY = 5cm
4. XZ = 17cm, XY = 15cm
5. YZ = 80cm, XZ = 100cm
6. XY = 25cm, XY = 20 cm

Answers

1) 5cm
2) 15m
3) 12cm
4) 8cm
5) 60cm
6) 15cm

17.2.1 PYTHAGOREAN TRIPLES

A Pythagorean triple consists of three positive integers such that $c^2 = a^2 + b^2$. The three integers 3, 4 and 5 is a well-known Pythagorean triple because $5^2 = 3^2 + 4^2$.

Worked Example [4]

Questions

Check if the following number are Pythagorean triples

a) 32, 42 and 50
b) 15, 30 and 35
c) 5, 12 and 13

Solutions

a) $32^2 + 42^2$

$= 1024 + 1764$

$= 2788$

$50^2 = 2500$

Therefore $32^2 + 42^2 \neq 50^2$

Hence 32, 42 and 50 **Not a Triple.**

b) $15^2 + 30^2$

$= 225 + 900$

$= 1125$

$35^2 = 1225$

Therefore $15^2 + 30^2 \neq 35^2$

Again 15, 30 and 35 **Not a Triple.**

c) $5^2 + 12^2$

$= 25 + 144$

$= 169$

$13^2 = 169$

Therefore $5^2 + 12^2 = 13^2$

Now 5, 12 and 13 **This is a Triple.**

Activity (*17.2*) Pythagorean triples

Questions

Check if the following whole numbers are Pythagorean Triples:

a) 5, 12 and 13.

b) 7, 24 and 25.

c) 10, 24 and 26.

d) 15, 22 and 27.

e) 9, 40 and 41.

Answers

a) Pythagoras Triple.

b) Pythagoras Triple.

c) Pythagoras Triple.

d) Not Pythagoras Triple.

e) Pythagoras Triple.

Tip

Pythagorean triples are always right-angled triangles and numbers which are not Pythagorean triples cannot give us right-angled triangles.

17.3 TRIGONOMETRIC RATIOS

There are three basic trigonometric ratios: sine, cosine and tangent. Each of these relates an angle of a right-angled triangle to a ratio of the lengths of two of its sides.

The sides of the triangle have names, two of which are dependent on their position in relation to a given angle.

The side opposite the right angle is called the hypotenuse. This is the longest side. The side opposite to the given angle is called the opposite side. The side which is next to the given angle is called the adjacent side.

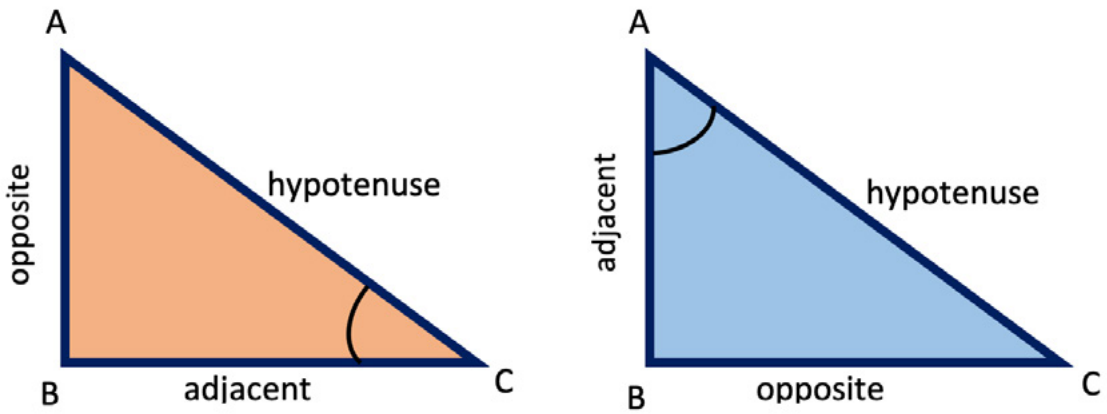


Fig 17.5 – Names of the sides of a triangle

17.3.1 Tangent

In a right angled triangle, the tangent of an angle is the length of the opposite side divided by the length of the adjacent side.

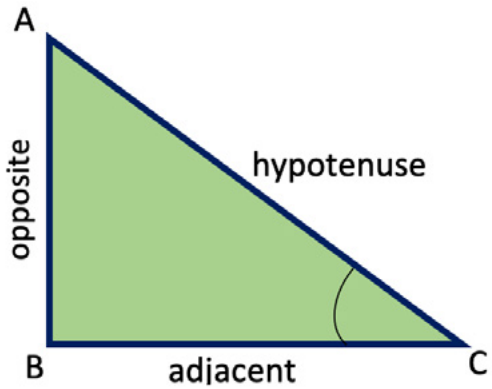


Fig 17.6 – Tangent of angle C

$$\tan C = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$

Worked Example [5]

Questions

Calculate the size of angle BCA in the following triangles

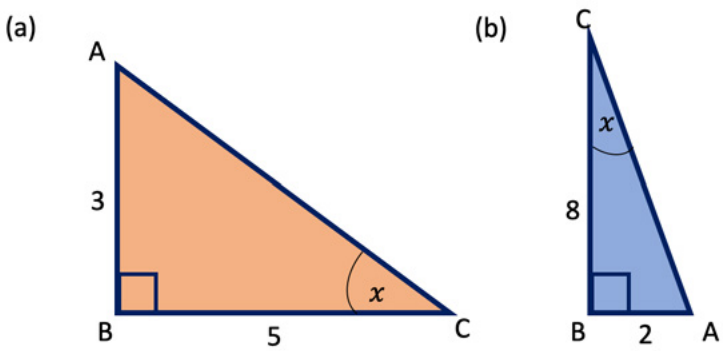


Fig 17.7

Solutions

a) $\tan x° = \frac{\text{opposite}}{\text{adjacent}}$

$\tan x° = \frac{3}{5}$

$x = \tan^{-1}(0.6)$

Since you are finding angle x, you have to set your calculator to Deg mode.

Use $\tan^{-1}$ on your calculator

$x = 30.9°$

$BCA = 31°$

b) $\tan x° = \frac{\text{opposite}}{\text{adjacent}}$

$\tan x° = \frac{2}{8}$

$x = \tan^{-1}(0.25)$

$x = 14.036°$

$BCA = 14°$

Worked Example [6]

Questions

1) Calculate the length of the opposite side on triangle (a)
2) Calculate the length of the adjacent side on triangle (b)

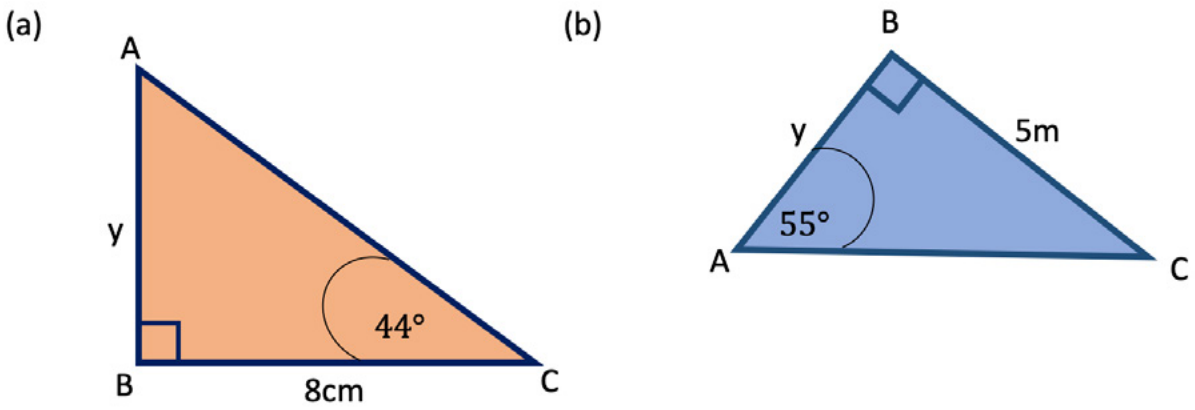


Fig 17.8

Solutions

1) $\tan x° = \frac{\text{opposite}}{\text{adjacent}}$

$\tan 44° = \frac{y}{8}$

$8 \times \tan 44° = y$

$y = 7.73$

2) $\tan x° = \frac{\text{opposite}}{\text{adjacent}}$

$\tan 55° = \frac{5}{y}$

$y \times \tan 55° = 5$

$y = \frac{5}{\tan 55°}$

$y = 3.5\text{m}$

17.3.2 Sine

In a right angled triangle, the sine of an angle is the length of the opposite side divided by the length of the hypotenuse side.

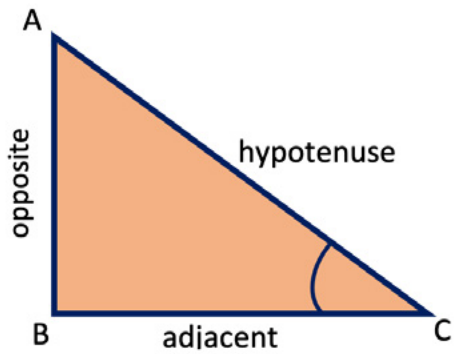


Fig 17.9 – Sine of angle C

$$\sin C = \frac{\text{length of opposite side}}{\text{length of hypotenuse side}}$$

Worked Example [7]

Questions

Calculate the size of angle BCA in the following triangles

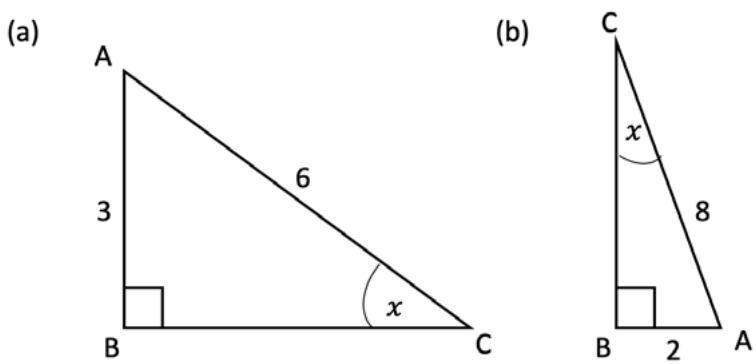


Fig 17.10

Solutions

a) $\sin x° = \frac{\text{opposite}}{\text{hypotenuse}}$

$\sin x° = \frac{3}{6}$

$x = \sin^{-1}(0.5)$

$x = 30°$

b) $\sin x° = \frac{\text{opposite}}{\text{hypotenuse}}$

$\sin x° = \frac{2}{8}$

$x° = \sin^{-1}(0.25)$

$x = 14.5°$

Worked Example [8]

Questions

1) Calculate the length of the hypotenuse on triangle (a)
2) Calculate the length of the opposite side on triangle (b)

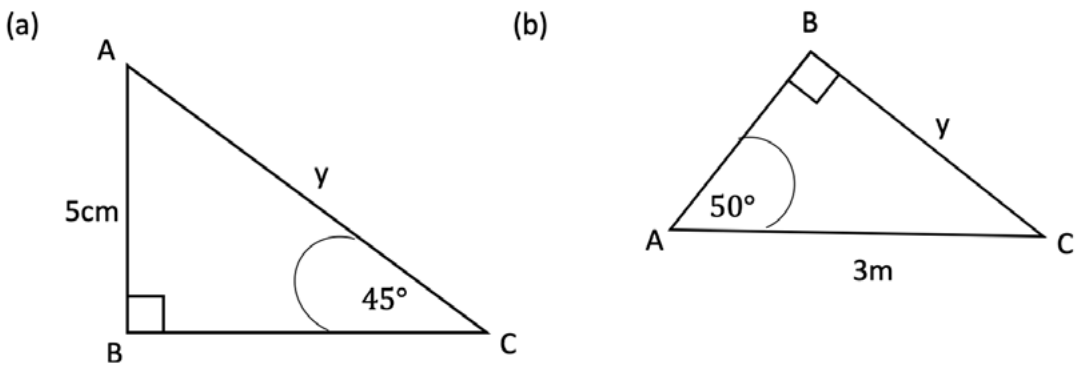


Fig 17.11

Solutions

1) $\sin x° = \frac{\text{opposite}}{\text{hypotenuse}}$

$\sin 45° = \frac{5}{y}$

$y \times \sin 45° = 5$

$y = \frac{5}{\sin 45°}$

$y = 7.07\text{cm}$

2) $\sin x° = \frac{\text{opposite}}{\text{hypotenuse}}$

$\sin 50° = \frac{y}{3}$

$3 \times \sin 50° = y$

$y = 2.3\text{m}$

17.3.3 Cosine

In a right angled triangle, the cosine of an angle is the length of the adjacent side divided by the length of the hypotenuse.

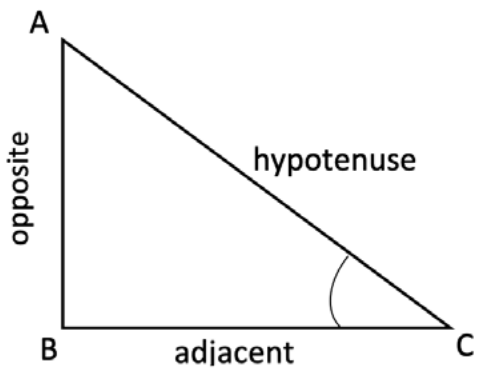


Fig 17.12 – Cosine of angle C

$$\cos C = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$

Worked Example [9]

Questions

Calculate the size of angle BCA in the following triangles

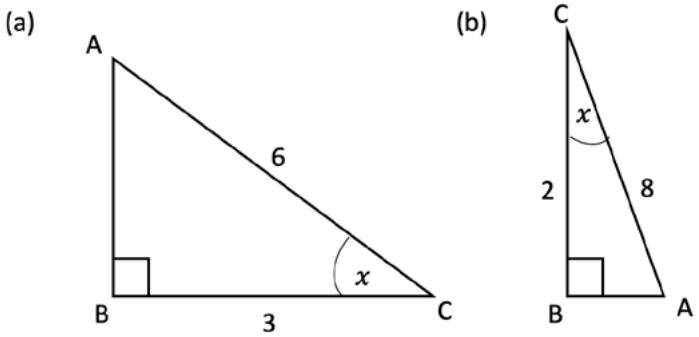


Fig 17.13

Solutions

a) $\cos x° = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\cos x° = \frac{3}{6}$

$x = \cos^{-1}(0.5)$

$x = 60°$

b) $\cos x° = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\cos x° = \frac{2}{8}$

$x = \cos^{-1}(0.25)$

$x = 75.5°$

Worked Example [10]

Questions

1) Calculate the length of the hypotenuse on triangle (a)
2) Calculate the length of the adjacent side on triangle (b)

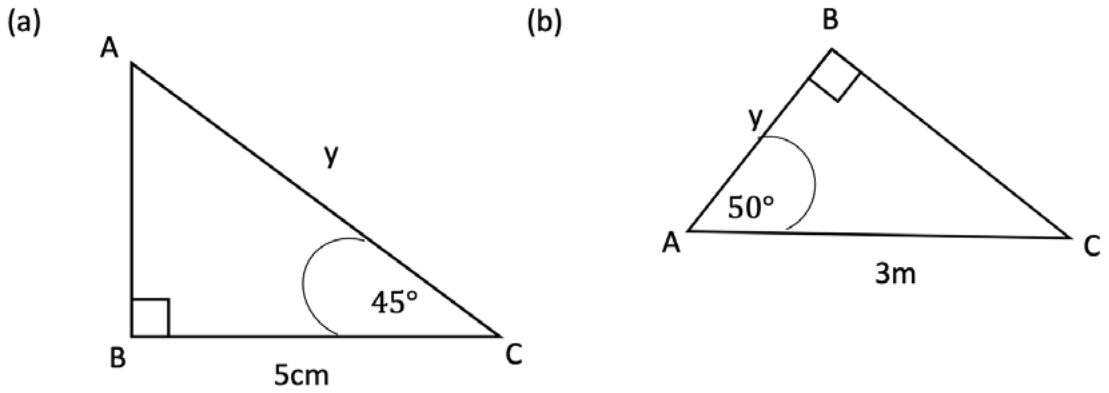


Fig 17.14

Solutions

1) $\cos x° = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\cos 45° = \frac{5}{y}$

$y \times \cos 45° = 5$

$y = \frac{5}{\cos 45°}$

$y = 7.07\text{cm}$

2) $\cos x° = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\cos 50° = \frac{y}{3}$

$3 \times \cos 50° = y$

$y = 1.93\text{m}$

Activity (*17.3*) Trigonometric ratios

Questions

1) Find the value of **a** in each of the triangles below:

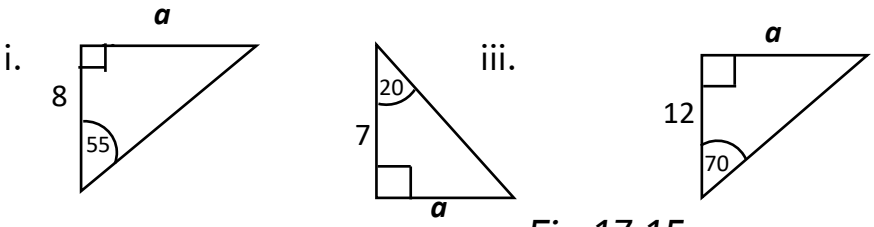


Fig 17.15

2) Calculate the value of **b** in each of the triangles below:

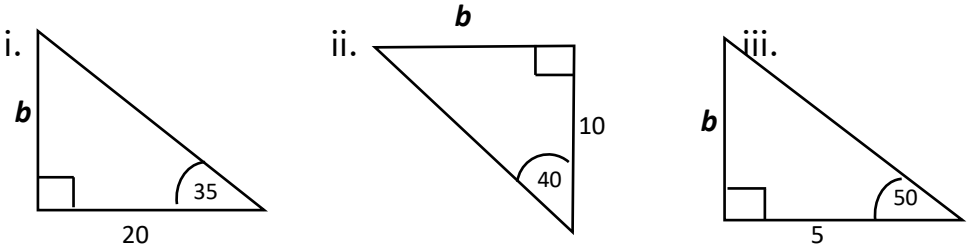


Fig 17.16

3) Find the value of **d** in each of the triangles below:

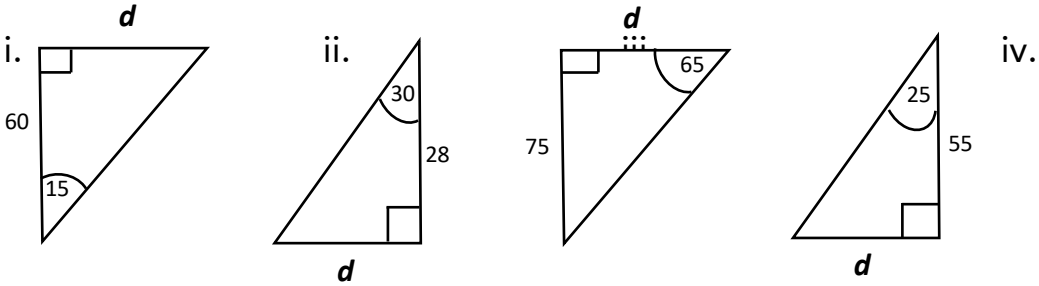


Fig 17.17

Answers

1. i. 11.4 ii. 2.55 iii. 33.0
2. i. 14.0 ii. 8.39 iii. 5.96
3. i. 16.1 ii. 16.2 iii. 35.0 iv. 25.6

The following activity requires the application of both Pythagoras' theorem and trigonometric ratios.

Activity (*17.4*) Finding missing angles and sides

Questions

Attempt as many questions as possible to solve right-angled triangles by applying Pythagoras theorem and trigonometric ratios. Your calculator will simplify your working.

(Tip sketch diagrams in most of the questions)

1. Show as many ways as you can on how you can solve the trigonometric ratios in the diagram belc

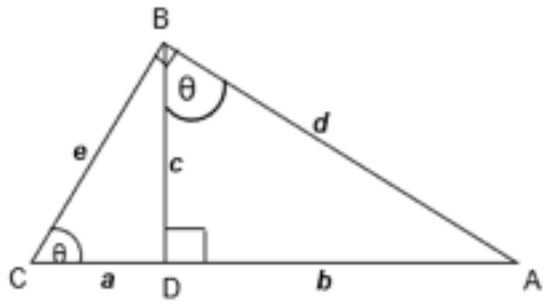


Fig 17.18

a) Tan **θ** b) Sin **θ** c) Cos **θ**

2. Use the diagram in question 1 above to calculate the sizes of the angles given that AB = 10, BD = 6 and CD = 8.

(Tip find missing sides first that is e and d)

3. If a boy walks for 600 metres from point T to point S and the 1000 metres from point S to point Q as shown in the diagram below.
   a) Calculate the angle between TS and SQ,
   b) Calculate his distance now from T.

(Tip use scale 1:100, remember to multiply your answer by 100)

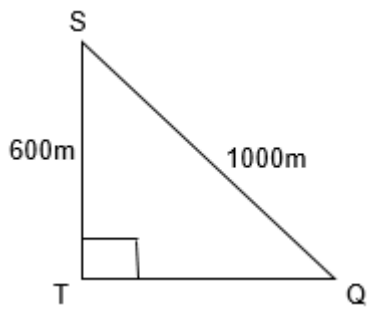


Fig 17.19

Answers

1) There many ways of solving the trig ratios.

   a) Let us start with Tangent

   $\tan\theta = \frac{c}{a}$ or $\tan\theta = \frac{b}{c}$ or $\tan\theta = \frac{e}{d}$

   b) Now we use Sine

   $\sin\theta = \frac{c}{e}$ or $\sin\theta = \frac{b}{d}$ or $\sin\theta = \frac{d}{a+d}$

   c) And lastly we use Cosine

   $\cos\theta = \frac{a}{e}$ or $\cos\theta = \frac{e}{a+b}$ or $\cos\theta = \frac{c}{d}$

2) To find the missing sides

   Side e

   $e^2 = a^2 + c^2$

   $e^2 = 8^2 + 6^2$

   $e^2 = 64 + 36$

   $e = \sqrt{100}$

   $e = 10$

Side b

$b^2 = d^2 - c^2$

$b^2 = 10^2 - 6^2$

$b^2 = 100 - 36$

$b = \sqrt{64}$

$b = 8$

Now let us find $\tan\theta$ for the first angle

$\tan\theta = \frac{6}{8}$

$= \tan^{-1}(0.75)$

$= 36.9°$

Now let us find $\tan\theta$ for the second angle

$\tan\theta = \frac{8}{6}$

$= \tan^{-1}(1.333)$

$= 53.3°$

Now let us find $\cos\theta$ for the first angle

$\cos\theta = \frac{8}{10}$

$= \cos^{-1}(0.8)$

$= 36.9°$

Now let us find $\cos\theta$ for the second angle

$\cos\theta = \frac{6}{10}$

$= \cos^{-1}(0.6)$

$= 53.3°$

Now let us find $\sin\theta$ for the first angle

$\sin\theta = \frac{6}{10}$

$= \sin^{-1}(0.6)$

$= 36.9°$

Now let us find $\sin\theta$ for the second angle

$$\sin\theta = \frac{8}{10}$$
$$= \sin^{-1}(0.8)$$
$$= 53.3°$$

3) Now let us find $\cos\theta$ for the angle TSQ and then the length of TQ

a) Finding $\cos\theta$ for the angle TSQ

$$\cos\theta = \frac{600}{1000}$$
$$\cos\theta = \frac{6}{10}$$
$$= \cos^{-1}(0.6)$$
$$= 53.3°$$

b) Finding the length of TQ

$$TQ^2 = 10^2 - 6^2$$
$$TQ^2 = 100 - 36$$
$$TQ = \sqrt{64}$$
$$TQ = 8$$
$$TQ = 8 \times 100 \text{ (since we used a scale of 1:100)}$$
$$TQ = 800 \text{ metres}$$

17.4 ANGLES OF ELEVATION AND DEPRESSION

The angle of elevation is the angle above the horizontal through which a line of view is raised. The angle of depression is the angle below the horizontal through which a line of view is lowered.

Worked Example [11]

Questions

The base of a pole is 80 metres from point X on the ground. If the angle of elevation of the top of the pole from X is 40. Calculate the height of the building.

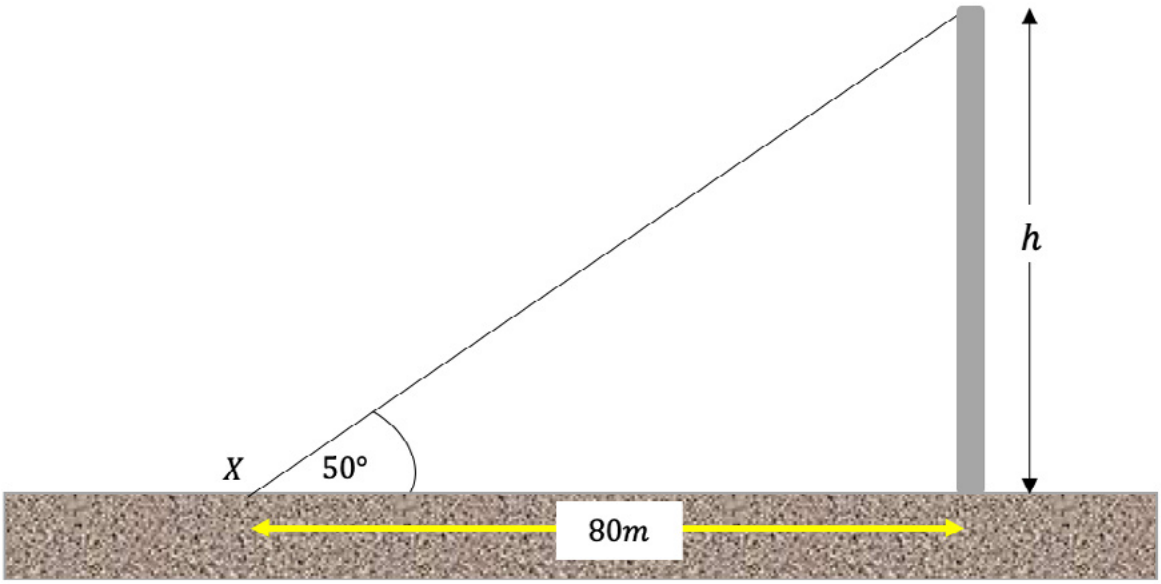


Fig 17.20

Solution

The shape produced here is a right angled triangle with adjacent and opposite sides to be used in the calculation. This means we should use tan to solve for the unknown height.

$$\tan x° = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 50° = \frac{h}{80}$$

$$80 \tan 50° = h$$

$$95.4\text{m} = h$$

Worked Example [12]

Questions

A raven is flying high up in the sky and it notices a mouse which is on the ground at point X. If the distance of the mouse to the raven is 60m and angle of depression of point X from the raven is 35°, calculate the height at which the raven is flying.

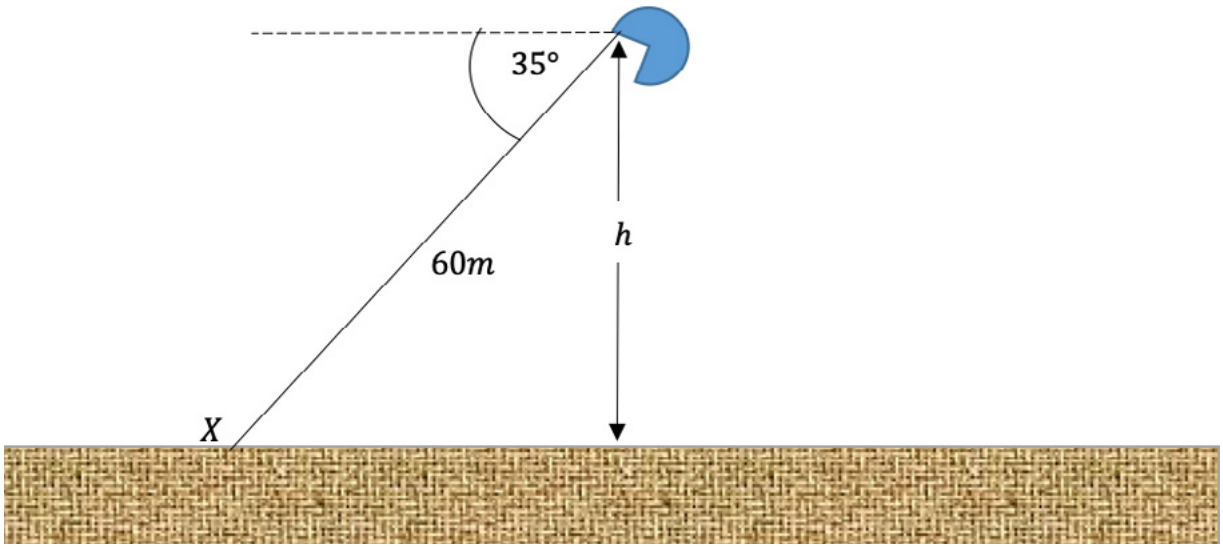


Fig 17.21

Solution

The shape produced here is a right angled triangle with hypotenuse and opposite sides to be used in the calculation. This means we should use sin to solve for the unknown height.

$$\sin x° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 35° = \frac{h}{60}$$

$$60 \tan 35° = h$$

$$34.4\text{m} = h$$

Activity (*17.5*) Angles of elevation and depression

Questions

1) A ladder 8m long lean against a wall and is 2m from the wall. Find how high up the wall the ladder reaches.
2) A river 600 meters deep and 700 meters wide. Water is to be drawn from the bottom of the river using a pipe. Now you are required to calculate the length of the pipe.
3) Using given information in the diagram below, calculate the length HG.

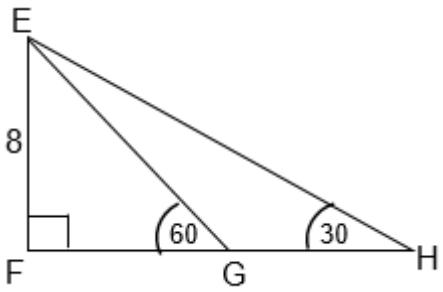


Fig 17.22

4) The angle of elevation to the top of a church is 25° from a point 80 meters away on level ground. Calculate how high the church is.
5) A tree is 75 meters tall, calculate the length of its shadow when the angle of elevation of the sun is 48°.
6) The diagonal of a room is 12 meters and the longer side is 8 meters.
   a) Calculate the angle between the diagonal and the shorter side.
   b) Find the width of the room.
7) A ladder is 5 meters long leans on a vertical wall and its foot is 4 meters away from the wall.
   a) Calculate the angle it makes with the floor.

(Tip give answer to 1 decimal point)

   b) Calculate the height of the wall.
8) The angle of depression from a window 12 meters up to a coin on the ground is 20°. Calculate the distance of the coin from the base of the house.

Answers

1. 8.94m.
2. 922m.
3. 9.24m.
4. 37.3m.
5. 67.5m.
6. a) 41.8°. b) x = 8.94.
7. a) 53.1° b) 3m.
8. 11.3m.

17.5 Summary

- Pythagoras' Theorem

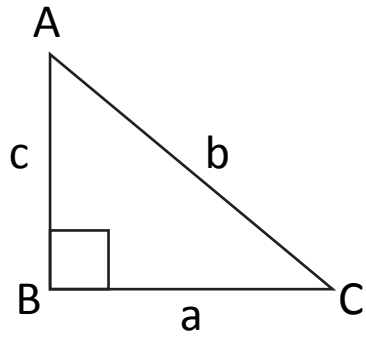


Fig 17.24

- For any right-angled triangle:
  $AB^2 + BC^2 = AC^2$
  i.e $c^2 + a^2 = b^2$ and $a^2 = b^2 - c^2$
- A set of three whole numbers (a, b, c) on a right-angled triangle are called Pythagoras triple and they obey Pythagoras theorem.
- Horizontal lines and surfaces are parallel to the surface of the earth. Vertical lines are perpendicular to any horizontal surface.

17.6 Further Reading

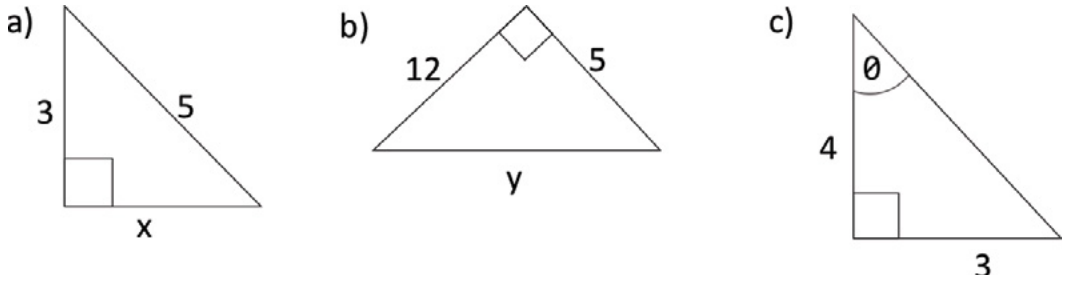
1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

17.7 Assessment Test

1) Which of the following are Pythagorean triples:

   a) (24, 58, 62)
   b) (50, 120, 30)
   c) (30, 40, 50)

2) Calculate x, y and θ in the following triangles.

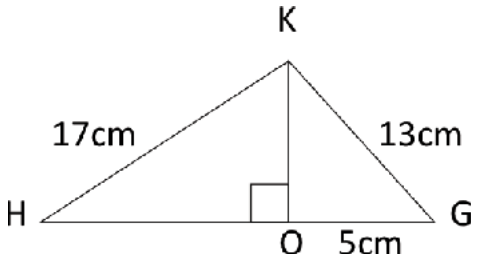


3) In a triangle XYZ, Y = 90° YZ = 8 cm X = 60°.

   a) Calculate angle Z.
   b) Calculate side XYZ.

4) Using the diagram below, calculate:

   a) Angle KGO.
   b) HG.
   c) The area of triangle HKG. (give your answers to 1 decimal place if not exact)



5) Triangle ABC is a right-angled triangle at B, angle C = 30° and side BC = 6 cm.

   a) Calculate angle A.
   b) Hence of otherwise calculate BA to 2 decimal places.

6) A ladder 8 meters long leans against a wall, its foot is 6 meters away from the base of a wall calculate:
   a) The angle formed between ground and the ladder.
   b) Height of the wall.

Answers

1. a) Not a triple	b) a triple	c) a triple.
2. a) x = 4	b) y = 13	c) 36.9 °.
3. a) 30°	b) 6.93	
4. a) 22.7 °	b) 12.0 °	c) 102 cm^2.
5. a) 60°	b) 3.46.	
6. a) 41.5°	b) 5.29m.	

UNIT 18 - VARIATION

CONTENTS

18.1 INTRODUCTION

Have you ever asked yourself how quantities or variables are related? Think of water bills and rates against consumption and try to figure out how these quantities are evaluated or charged. In this unit we are simply going to explore how quantities are directly, inversely, jointly and partially related.

Objectives

After going through this unit, you should be able to:

- Express the types of variation in algebraic terms
- Distinguish between types of variation
- Sketch graphs to illustrate direct and inverse variation
- Solve problems involving variation

Key words

$\propto$ **-** is a variation sign which joins 2 or more quantities which are proportional to each other.

Direct variation – is the variation which includes 2 quantities varying directly to each other.

Inverse variation –is the variation which involves 2 quantities that vary indirectly to each other that is as one quantity increases the other decreases and vice versa.

Joint variation- is the variation which involves 3 or more quantities joined together with a multiplicative rule.

Partial variation- is the variation that occurs when one quantity has a fixed value and at the same time it is also proportional to another variable.

Time You are advised not to take more than 8 hours in this unit.

Study Skills

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems on Variation as possible for you to grasp all the concepts in this topic. Revisit the topic on Simultaneous Equations before you begin this topic

18.2 DIRECT VARIATION

It is the variation which includes 2 quantities varying directly to each other. This means that if one quantity increases the other quantity will also increase and as well if one quantity decreases the other quantity will also decreases.

Worked Example [1]

Questions

2 quantities $y\ and\ x$ are related such that y varies directly as x. When $y = 9, x = 45$.

a) Write down the relationship connecting $x, y\ and\ k$.
b) Find the value of k.
c) Write the formula connecting $y\ and\ x$.
d) Find y when $x = 40$
e) Find x when $y = 10$.

Solutions

a) For us to tackle this question we have to write the statement in a mathematical way

   $y \propto x$,in order to remove the variation sign introduce a constant

   $y = kx$.

b) For us to find the value of k we have to substitute the values of y and x in the equation above.

$y = kx$, but $y = 9$ and $x = 45$.

$9 = 45k$,

$\therefore k = \frac{1}{5}$.

c) In order to find the formula connecting y and x, we have to substitute the value of k in the equation in (a) above.

$y = kx$, but $k = \frac{1}{5}$

$\therefore y = \frac{1}{5}x$.

d) In order to find the value of y given the value of x, we first write the equation involving y and x and then substitute the values in the equation.

$y = \frac{1}{5}x$, but $x = 40$

$y = \frac{1}{5}(40)$,

$\therefore y = 8$.

e) Here, we are just repeating what we did in question (d) above

$y = \frac{1}{5}x$, but $y = 10$

$10 = \frac{1}{5}x$, by multiplying both sides by 5 we get;

$x = 50$.

Worked Example [2]

Questions

Given that the mass of an object is directly proportional to its weight and its mass (m) is 30g when its weight (w) is 6.

a) Find the equation connecting m and w and a constant.

b) Express m in terms of w.

c) Find m, when $w = 1\frac{2}{5}$

Solutions

a) $m \propto w$

$m = kw$, where k is a constant

b) Since the question is asking to express m in terms of w, first we need to find the value of the constant k.

$m = kw$, but $m = 30$ and $w = 6$.

$30 = 6k$,

$\therefore k = 5$.

Now, expressing m in terms with $k = 5$ we get;

$m = 5w$.

c) $m = 5w$, but $w = 1\frac{2}{5}$,

$m = 5(\frac{7}{5})$,

$\therefore m = 7$.

Worked Example [3]

Questions

If y varies directly as the cube of x and $y = 40$ when $x = 2$.

Find the value of y when $x = 3$.

Solution

Following the steps in the examples above we get;

$y \propto x^3$, by introducing a constant we get;

$y = kx^3$,where k is a constant

Substituting values of y and x we get;

$40 = 2^3k$,

$40 = 8k$,

$k = 5$.

Expressing y in terms of we get;

$y = 5x^3$,

Now, finding the value of y when $x = 3$.

$y = 5(3^3)$,

$y = 5 \times 27$,

$\therefore y = 135$.

Activity (*18.1*) Direct Variation

Questions

By referring to the above examples try to attempt the following questions.

1) m is directly proportional to n^2 and $m = 28$, when $n = 2$.
   a) Write down an expression for m in terms of n and constant k.
   b) Calculate the value of k.
   c) Calculate the value of n when $m = 63$.

2) M is directly as the square of $(d - 1)$. Given that $M = 12$, when $d = 4$.
   a) express M in terms of d.
   b) Calculate M when $d = 7$.

3) x varies directly as the square of y. If $x = 9$, when $y = 3$, find
   a) the relationship between x, y and a constant k.
   b) the value of k.
   c) the equation in x and y.
   d) the value of x when $y = \frac{1}{2}$.

Answers

1a) $m = kn^2$ b) $k = 7$ $c)\ n = \pm 3$

2a) $m = \frac{4}{3}(d-1)^2$ b) M = 48

3a) $x = ky^2$ $b)\ k = 1$ $c)\ x = y^2$ $d)\ x = \frac{1}{4}$

18.3 INVERSE VARIATION/ INDIRECT VARIATION

The proportion is opposite to the direct variation.

When one quantity increases the other quantity decreases and vice versa.

Worked Example [4]

Questions

It is given that y is indirectly proportional to x.when y is 4, $x = 2$.

a) Express y in terms of x and a constant k.

b) Find the value of the constant k.

c) Write down a formula connecting y and x.

d) Find y when $x = 16$.

Solution

a) If y is indirectly proportional to x, it means that y is proportional to the inverse or reciprocal of x.but the inverse of x is $\frac{1}{x}$ so;

$y \propto \frac{1}{x}$,we now introduce the constant inorder to remove the variation sign so,

$y = k \times \frac{1}{x}$,

$y = \frac{k}{x}$.

b) $y = \frac{k}{x}$, but $y = 4$ and $x = 2$

$4 = \frac{k}{2}$

$\therefore k = 8$.

c) $y = \frac{k}{x}$, but $k = 8$

$\therefore y = \frac{8}{x}$.

d) $y = \frac{8}{x}$, but $x = 16$

$y = \frac{8}{16}$,

$\therefore y = \frac{1}{2}$.

Worked Example [5]

Questions

It is given that a varies inversely as the square root of b and that when $a = 2, b = 9$.

a) express a in terms of b and constant k.

b) find the value of k.

c) write an equation in a and b.

d) find b when $a = \frac{1}{2}$.

Solutions

By following the steps as in the example above, lets solve the questions.

a) $a \propto \frac{1}{\sqrt{b}}$

$a = k(\frac{1}{\sqrt{b}})$,

$a = \frac{k}{\sqrt{b}}$.

b) $a = \frac{k}{\sqrt{b}}$, but $a = 2$ and $b = 9$

$2 = \frac{k}{\sqrt{9}}$,

$2 = \frac{k}{3}$, by clearing the denominator we get;

$\therefore k = 6$.

c) $a = \frac{k}{\sqrt{b}}$, but $k = 6$

$\therefore a = \frac{6}{\sqrt{b}}$.

d) $a = \frac{6}{\sqrt{b}}$, but $a = \frac{1}{2}$

$\frac{1}{2} = \frac{6}{\sqrt{b}}$, by cross multiplication we get;

$\sqrt{b} = 12$, by squaring both sides we get;

$\sqrt{b^2} = 12^2$, simplifying both sides we get;

$b = 144$.

Worked Example [6]

Questions

If y is indirectly proportional to the square of $(x+3)$ and $y = 2$ when $x = 4$. Find the formula connecting x and y.

Solution

$y \propto \frac{1}{(x+3)^2}$, by introducing a constant k we get;

$y = k(\frac{1}{(x+3)^2})$,

$y = \frac{k}{(x+3)^2}$, but $y = 2$ and $x = 4$ so;

$2 = \frac{k}{(4+3)^2}$,7 by clearing the denominator we get;

$2(49) = k$,

$\therefore k = 98$, now substituting the value of k in the formula we get;

$y = \frac{98}{(x+3)^2}$.

Activity (*18.2*) Inverse variation

Questions

1. y is inversely proportional to the cube of x and when $y = 60, x = 2$.
   a) Express x, in terms of y and$k$.
   b) Find the value of k.
   c) Write down an equation in x and y.
   d) Find x when $y = 100$,giving your answer to 2 decimal places.
2. It is given that d varies inversely as the square of $(e-1)$. When $d = 2, e = 2$.
   a) write down a formula connecting d, e and constant m.
   b) find the value of m.
   c) what is the relationship between d and e?
   d) find the value of d when $e = 4$.
3. It is given that r is indirectly proportional to $(2m+3)$ and that when $r = 1, m = 1$.
   a) Express r in terms in terms of m.
   b) Find the value of m when $r = 5$.

Answers

1 a) $y = \frac{k}{x^3}$ $b)\ k = 480$ $c)\ y = \frac{480}{x^3}$ $d)\ x = 4{,}80$

2 a) $d = \frac{m}{(e-1)^2}$ $b)\ m = 2$ $c)\ d = \frac{2}{(e-1)^2}$

3 a) $r = \frac{5}{2m+3}$ $b)\ m = -1$

18.4 JOINT VARIATION

Joint variation is the variation which involves 3 or more quantities joined together with a multiplicative rule.

Worked Example [7]

Questions

It is given that d varies jointly as e^2 and f. If $\mathrm{f} = 2, \mathrm{e} = 3$ when $\mathrm{d} = 5$, find

a) the formua connecting $\mathrm{d}, \mathrm{e}, \mathrm{f}$ and constant k.

b) find the value of k.

c) the equation for d in terms of e and f.

d) the value of d when $\mathrm{e} = 2$ and $\mathrm{f} = 3$.

Solution

a) $\mathrm{d} \propto \mathrm{e}^2\mathrm{f}$,

$\mathrm{d} = \mathrm{ke}^2\mathrm{f}$.

b) $\mathrm{d} = \mathrm{ke}^2\mathrm{f}$, but $\mathrm{d} = 5, \mathrm{e} = 3, \mathrm{f} = 2$. Sustituting values we get;

$5 = 3^2(2)\mathrm{k}$,

$5 = 18\mathrm{k}$, dividing throughout by 18 we get;

$\mathrm{k} = \frac{5}{18}$.

c) $\mathrm{d} = \mathrm{ke}^2\mathrm{f}$, but $\mathrm{k} = \frac{5}{18}$

$\therefore \mathrm{d} = \frac{5}{18}\mathrm{e}^2\mathrm{f}$.

d) $d = \frac{5}{18}e^2f$,but $e = 2$ and $f = 3$.

$d = \frac{5}{18} \times 2^2 \times 3$,

$\therefore d = 1\frac{2}{3}$.

Worked Example [8]

Questions

Given that T varies directly as x and inversely as $(y + 4)$.

a) write down an expression connecting T, x, y and constant k.

b) find the value of k if T=5 when $x = 2$and $y = 14$.

c) express T in terms of x and y.

Solutions

a) $T \propto x$ and $T \propto \frac{1}{(y+4)}$, joining the 2 variations by multiplication we get;

$T \propto \frac{x}{(y+4)}$, now introducing a constant we get;

$T = k \times \frac{x}{(y+4)}$

$T = \frac{kx}{(y+4)}$.

b) $T = \frac{kx}{(y+4)}$, now sustituting the values of T, x, y we get;

$5 = \frac{2k}{14+4}$

$5 = \frac{2k}{18}$, multiplying both sides by 18 we get;

$80 = 2k$, dividing both sides by 2 we get;

$k = 40$.

c) $T = \frac{kx}{(y+4)}$,but $k = 40$

$\therefore T = \frac{40x}{(y+4)}$.

Worked Example [9]

Questions

The annual premium, \$P, for a funeral insurance scheme varies jointly as the square of the age Y, years and the number of dependents, D of the applicant. If a 25-year old applicant with 6 dependents pays \$150, calculate the monthly premium for a 49-year old applicant with 4 dependents.

Solutions

First we have to write the above statement mathematically

$P \propto Y^2D$, now introducing a constant we get;

$P=kY^2D$, substituting the given values we get;

$150=25^2 \times 6k$, simplifying we get;

$150=3750k$, dividing both side by 3750 we get;

$k = \frac{1}{25}$, substituting the value of k in the equation we get;

$P=\frac{1}{25}Y^2D$,substituting the given values of Y nad D we get;

$P=\frac{1}{25} \times 49^2 \times 4$

$\therefore P=\$384,16$.

Activity (*18.3*) Joint variation

Questions

1. It is given that x varies jointly as the square root of y and t. If $t = 2, y = 9$ when $x = 1$, find
   a) the formula connecting x, y , t and constant k.
   b) find the value of k.
   c) the equation for x in terms of y and t.
   d) the value of x when $y = 25$ and $t = 3$.

2. Given that A varies directly as b and inversely as $2h$.
   a) write down an expression connecting A, b, h and constant k.
   b) find the value of k if A=10 when $b = 4$and $h = 7$.
   c) express A in terms of b and h.
   d) find the value of A if $b = 8$ and $h = 24$,

Answers

1 a) $x = ky^{\frac{1}{2}}t$ b) $k = \frac{1}{6}$ c) $x = \frac{1}{6}\, y^{\frac{1}{2}}\, t$ d) $x = 12.5$

2 a) A $= \frac{kb}{2h}$ b) $k = 3{,}5$ c) A $= \frac{35b}{2h}$ d) $A = 5\frac{5}{6}$

18.5 PARTIAL VARIATION

This variation occurs when one quantity is has a fixed value and at the same time it is also proportional another variable. The 2 parts of a fixed value and a variable are added together.

Worked Example [10]

Questions

The price P of a commodity varies partly as the demand, D and partly as the availability A.

a) Write down the expression for P in terms of D and A using the constants c and k respectively.

b) Given that P=19 ,D=5 when A=2 and that P=17,D=3 when A=4. Write down the 2 equations and solve them simultaneously to get the value of c and k.

c) Write down equation connecting P, D and A.

Solutions

a) P=Dc + Ak

b) 19=5c + 2k…..(1)

17=3c + 4k …..(2), solving the equations using elimination method,

First multiply equation (1) by 2 and equation(2) by 1

38=10c + 4k…..(3)

17=3c + 4k…….(4)

Subtracting equation(4) from equation(3) we get;

21=7c, dividing both sides by 7 we get;

c = 3.

Substituting the value of c in equation(1) we get;

38=10(3) +4k

$\therefore k = 8.$

c) We take equation from solution(a) above

P=D $c + Ak$, but $c = 3$ and $k = 8$.

P=3 $c + 8k$

Worked Example [11]

Questions

Rudo's weekly wage, W is partly constant and partly varies as the number of hours N of overtime she works per week.

a) express W in terms of N and constants h and k.

b) given that when W = 80, N = 10 and when W = 60, N = 6. Find the value of h and k.

c) what is the relationship between the Rudo's weekly wage and the overtime hours she works?

Solution

a) W=$h + Nk$

b) b)$80 = h + 10k$.....(1)

$60 = h + 6k$.....(2)

Subtracting equation (2) from equation(1) we get;

$20 = 4k$, dividing both sides by 4 we get;

$k = 5.$

Substituting the value of k in equation (1)we get;

$80 = h + 50,$

$\therefore h = 30.$

c) W = $h + Nk$, substituting the values of the constants we get;

W = 30 + 5N.

Worked Example [12]

Questions

It is given that y is partly constant and partly varies as the square of x.

a) a)express y in terms of x and constants h and k.

b) b)given that when $y = 35, x = 5$ and when $y = 27, x = -3$,find the value of h and k.

c) find the value of y when $x = -1$.

Solutions

a) $y = h + kx^2$

b) $35 = h + 25k$.....(1)

$27 = h + 9k$.....(2)

Subtracting equation(2) from (1) we get;

$8 = 16k$,

$\therefore k = \frac{1}{2}$.

Substituting the value of k in equation (2) we get;

$27 = h + \frac{9}{2}$,

$\therefore h = 22\frac{1}{2}$.

c) $y = h + kx^2$ but $k = \frac{1}{2}$ and $h = 22\frac{1}{2}$.

$y = 22\frac{1}{2} + \frac{1}{2}$,

$\therefore y = 23$.

Activity (*18.4*) Partial variation

Questions

1. It is given that y is partly constant and partly varies directly as x
   a) express y in terms of x, c and k where c and k are constants.
   b) given that $y = 6, x = 2$ and$y = -2$ when $x = 6$, find an expression for y in terms of x.
2. p varies partly as r and partly as inversely as t. when $r = 2, t = 8$ and $p = 2$. When $r = 2, t = 12$ and $p = 4$
   a) express p in terms of r, t and constants h and k.

b) find the value of h and k.

c) write down the formula connecting p, r and t.

Answers

1 a) $y = c + kx$ b) $y = 10 - 2x$

2 a) $p = rb + \frac{k}{t}$ b) $h = 4 \ k = -48$ c) $p = 4r - \frac{48}{t}$

18.6 Summary

- There are 4 types of variation namely, direct, indirect, joint and partial variation.
- All the variations have one constant of variation except for the partial variation which has 2 constants.
- In all the other 3 variations which are not partial, in order for you to introduce to remove the variation sign you have to introduce a constant.
- When solving partial variation questions, solve the equations formed simultaneously.

18.7 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

18.8 Assessment Test

1. It is given that x varies inversely as the square of $(y + 2)$. When $x = 4, y = 2$.

   a) write down a formula connecting x , y and constant k.

   b) find the value of k.

2. z varies directly as the square of w. If $z = 36$, when $w = 12$, find
   a) the relationship between z, w and a constant k.
   b) the value of k.
   c) the equation in z and w.
3. It is given that x varies jointly as the of n and m If $m = 2, n = 6$ when $x = 2$, find
   a) the formua connecting x, n, m and constant c.
   b) find the value of c.
   c) the equation for x in terms of n and m.
   d) the value of x when $m = 4$ and $n = 3$.

UNIT 19 - TRIGONOMETRY 2

CONTENTS

19.1 INTRODUCTION

You have noted that in Unit 17 you have dealt with one type of a triangle which is right-angled. Now in this unit we are going to learn more about other triangles which are not right angled. We are also going to simplify the concept by looking at some real life situations on bearing by learning sine, cosine rules and the area of a triangle. In this unit you need to have a scientific calculator.

OBJECTIVES

After going through this unit, you should be able to

- calculate the area of a triangle where the height is not given
- state and apply appropriately the sine and cosine rules
- draw diagrams representing bearings or word problems on sine rule and cosine rule

KEY TERMS

Bearing :– is a direction given as the number of degrees

Included angle: – an angle between two given sides of a triangle

TIME: You are required to spend at most 8 hours on this unit.

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems on Trigonometry as possible for you to grasp all the concepts in this topic. Reset your scientific calculator to indicate degrees (Deg) before attempting any question in this unit

19.2 AREA OF A TRIANGLE USING THE SINE RATIO

Consider △ ABC, where the height is not given

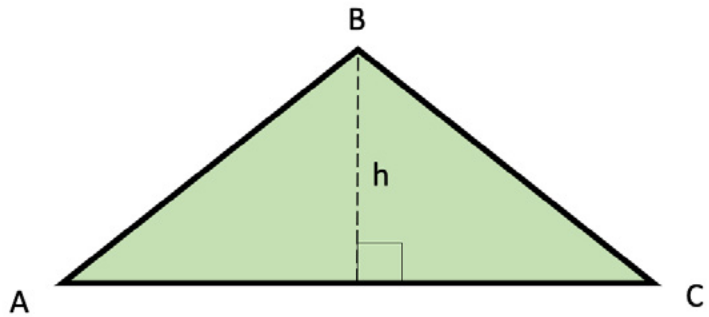


Fig 19.1

To find the area you may need to first find the height using the Sine ratio,

$$\text{Sin}A = \frac{\text{opp}}{\text{hyp}}$$

$$\text{Sin}A = \frac{h}{AB} \text{ [Hint make h the subject]}$$

$$h = AB \times \text{Sin}A$$

therefore, Area of triangle

$$= \frac{1}{2} \times b \times h$$

$$= \frac{1}{2} \times AC \times (AB \times \text{Sin}A)$$

Such triangles can also be presented as shown below.

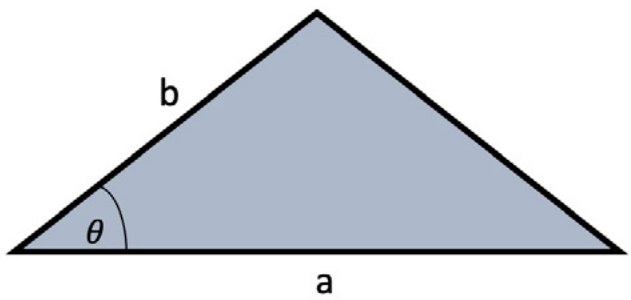


Fig 19.2

In short Area of triangle = $\frac{1}{2}$ab Sinθ

Worked Example [1]

Question

Find the area of triangle ABC, if AB = 7cm, AC = 16cm and the included angle $B\hat{A}C = 30^{o}$

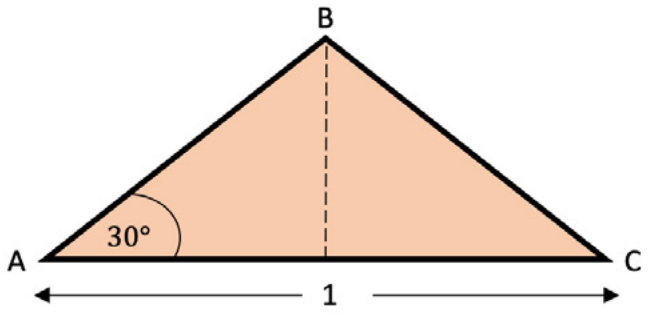


Fig 19.3

Solution

Let the height be h cm, such that

$\text{Sin}30^{o} = \frac{h}{7}$

$h = 7 \times \text{Sin}30^{o}$ [this is usually written as $h = 7\text{Sin}30^{o}$]

$h = 7 \times 0.5$ [Hint: use a scientific calculator]

$h = 3{,}5$ cm

then, Area of triangle = $\frac{1}{2} \times 16 \times 3{,}5$

Area = $8 \times 3{,}5$

hence, Area = 28 cm^2

Worked Example [2]

Questions

Find the area of the triangle below

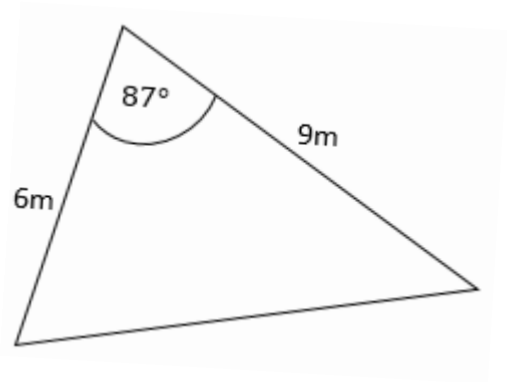


Fig 19.4

Solution

Area = $\frac{1}{2} \times ab \times \text{Sin}\theta$

Area = $\frac{1}{2} \times 6 \times 9 \times \text{Sin } 87^{o}$ [Hint: use a scientific calculator]

Area = 26,96299744

Area = 27,0 m^2 (3.s.f)

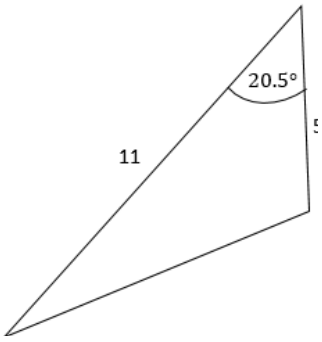
Now that we have done some examples on the area, attempt the following activity.

Activity (*19.1*) Area of a triangle

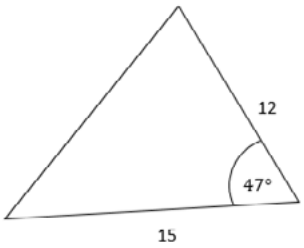
Questions

Find the areas of the following triangles, giving your answers correct to 3 significant figures. All lengths are given in centimetres (cm).

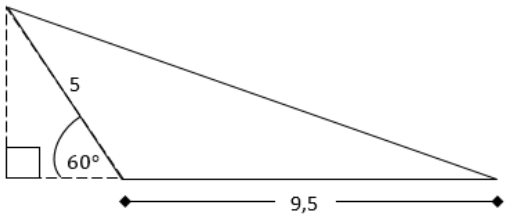
a)



b)



c)



d)

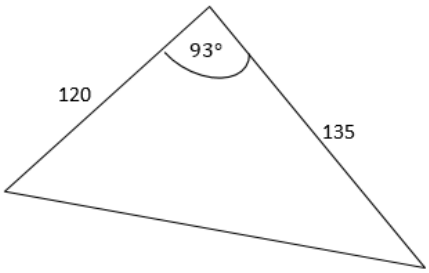


Fig 19.5

Answers

a) 9.63 b) 65.8

c) 20.6 d) 8089

19.3 THE SINE RULE

The Sine rule is a formula or a regulation that is used to solve for sides and angles of triangles which are not right-angled.

The Sine Rule applies where:-

(a) Two angles and any side of the triangle are given, say

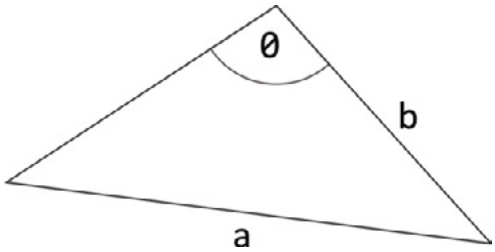


Fig 19.6

(b) Two sides and an angle opposite one of the sides of the triangle are given, say

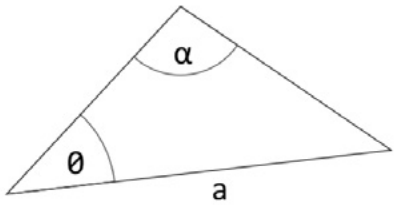


Fig 19.7

Tip: The Sine rule applies where there is a complete pair of a side and an angle given.

Given below is model triangle and the sine rule formulae for sides and angles.

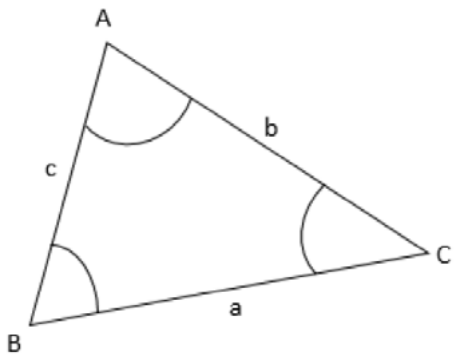


Fig 19.8

Formulae for sides

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$ a complete pair

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Formulae for angles

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$ a complete pair

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

Tips:

- the formulae for sides and angles are used in pairs as indicated above.
- the selection of formulae to apply will depend on the information provided in the question, as illustrated below.

Worked Example [3]

Questions

In triangle ABC below, AB = 9cm, $A\hat{B}C = 80^o$ and $A\hat{C}B = 30^o$.

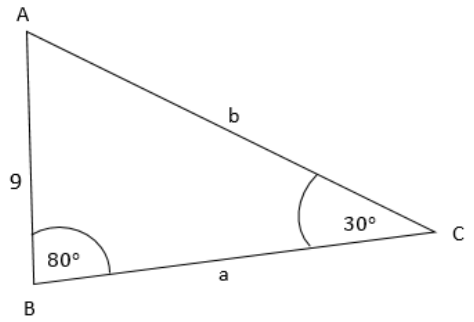


Fig 19.9

a) Write down the formula for finding side **b**

b) Hence, find side **b**

Solutions

a) $\frac{b}{\sin B} = \frac{c}{\sin C}$

b) $\frac{b}{\sin 80^o} = \frac{9}{\sin 30^o}$

$$b = \frac{9 \times \sin 80^o}{\sin 30^o}$$

$$b = \frac{9 \times 0.984807}{0.5}$$

$$a = 17{,}726526$$

$$a = 17{,}7 \text{ cm} \qquad (3 \text{ s.f})$$

Now, let us consider another example with more calculations and that should clarify the use of degrees and minutes as well.

Worked Example [4]

Questions

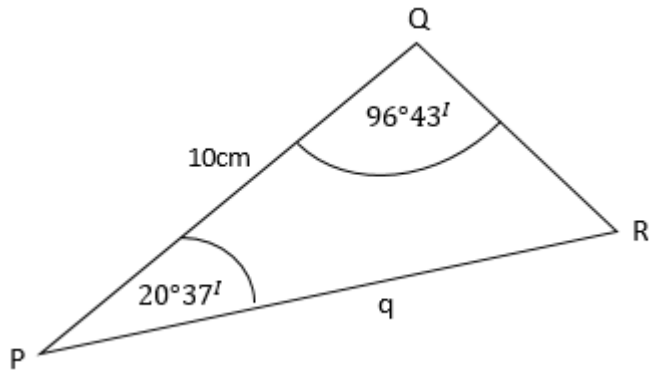


Fig 19.10

In the triangle PQR above, PQ = 10 cm, $P\hat{Q}R$ = 96°43′ and $Q\hat{P}R$ = 27°37′

Find.

a) $Q\hat{R}P$

b) Find side **q**

Solutions

a) $Q\hat{R}P$

= 180 – (96°43′ + 27°37′)

= 55°40′ Hint:-recall degrees and minutes in unit 17]

b) $\frac{q}{\sin 96^{o}43^{I}} = \frac{10}{\sin 55^{o}40^{I}}$

$q = \frac{10 \times \sin 96^{o}43^{I}}{\text{Sin } 55^{o}40^{I}}$ [Hint: use a scientific calculator]

$q = 12.02679073$

$q = 12.0$ (3.s.f)

Also consider an example that will help you in finding angles using the Sine Rule.

Worked Example [5]

Questions

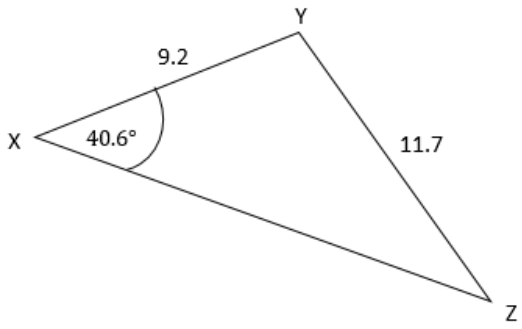


Fig 19.11

In triangle XYZ, XY = 9.2m, YZ = 11.7m and $Y\hat{X}Z = 40.6°$. Find

(i) State the correct formula for finding $X\hat{Z}Y$

(ii) Find $X\hat{Z}Y$

Solutions

(i) $\frac{\text{Sin } \hat{Z}}{z} = \frac{\text{Sin } X}{x}$

(ii) $\frac{\text{Sin } \hat{Z}}{9.2} = \frac{\text{Sin } 40.6^0}{11.7}$ this represents a given complete pair from the question

$\text{Sin } \hat{Z} = \frac{9.2 \times \text{Sin } 40.6^0}{11.7}$ [Hint: For easy computation let $\hat{Z} = X\hat{Z}Y$]

$\text{Sin } \hat{Z} = 0.5011719898$

$\hat{Z} = \text{Sin}^{-1}(0.5011719898)$

$\hat{Z} = 30.778445929^o$

Hence, $X\hat{Z}Y = 30.8^o$ (1 decimal place)

Activity (*19.1*)

Questions

1) Referring to example 3 and 4 above, find the lettered sides in the given triangles. All dimensions are given in metres (m). Also give your answers correct to 3 significant figures.

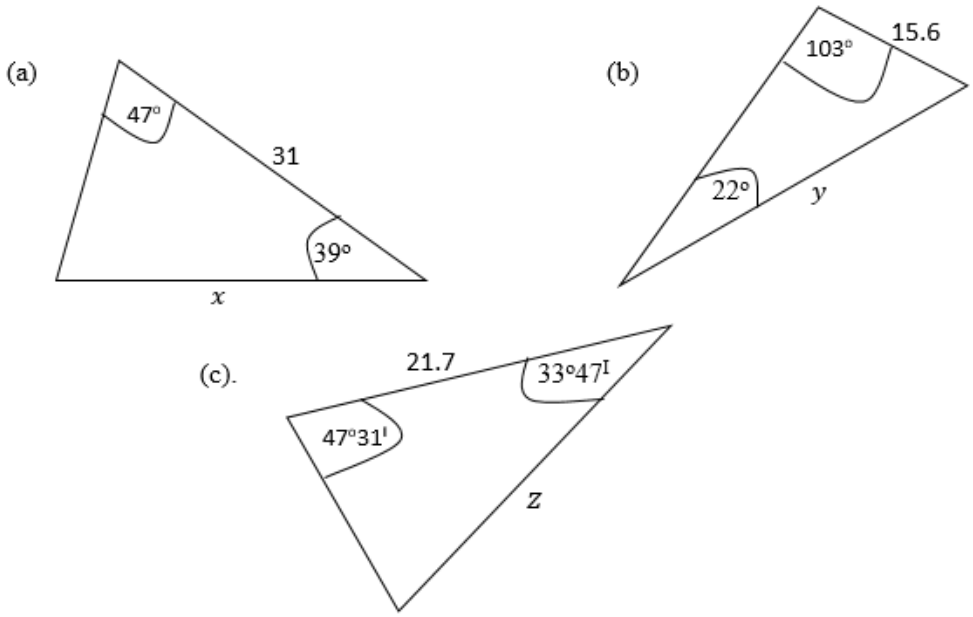


Fig 19.12

2) Referring to example 5 above, find the angles θ in the given triangles. All dimensions are given in metres (cm). Also give your answers correct to 1 decimal place.

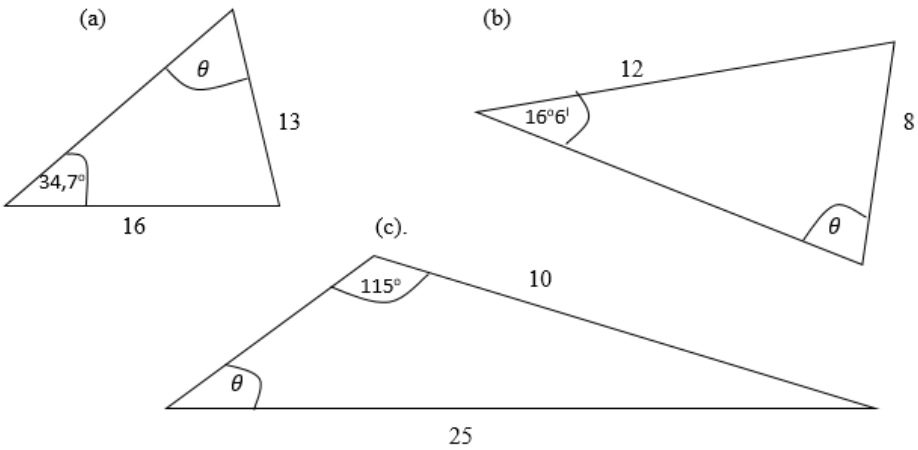


Fig 19.13

Answers

1. **(a)** $x = 22.7$ **(b)** $y = 40.6$ **(c)** $z = 16.2$
2. **(a)** $\theta = 44.5°$ **(b)** $\theta = 24.6°$ **(c)** $\theta = 21.3°$

19.4 SINE RULE WITH BEARING

Now that you able to solve for sides and angles using the sine rule, let's move a step further into the application of the same concept on bearing.

Tip: for each bearing question you should understand the diagrammatical representation of the question so that you can use the calculation learnt in the examples discussed earlier.

Worked Example [6]

Questions

A grazing cow moves from point **A** to point **B** a distance of 10m on a bearing of 025°, then changes direction and moves on a bearing of 120° towards a maize field at point **C**. If the initial position **A** of the cow is due west of the maize field **C**, find

(a) The distance between the initial position **A** of the cow and the maize field **C**.

(b) Find **BC.**

(c) Find the shortest distance from **B** to **AC**.

(d) State bearing of **B** from **C.**

Solutions

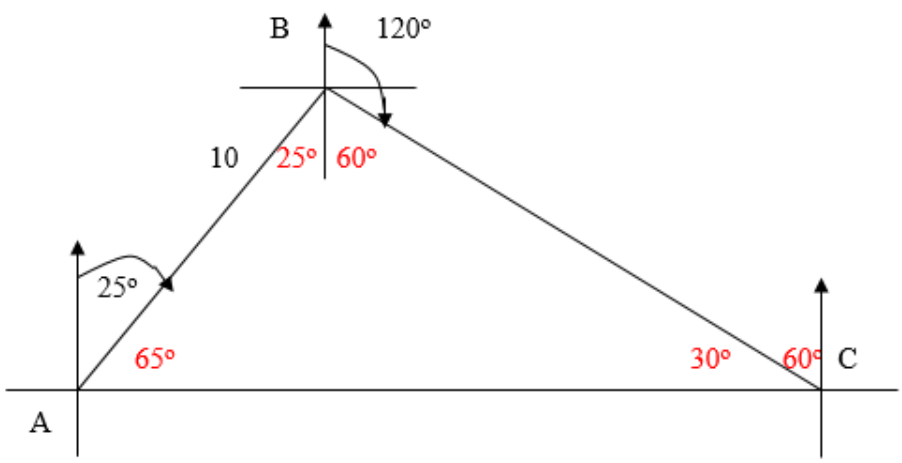


Fig 19.14

Tip: always represent questions of this type diagrammatically

(a) Using the Sine Rule

$$\frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}$$

$$\frac{AC}{\text{Sin } 85^0} = \frac{10}{\sin 30°}$$

$$AC = \frac{10 \times \text{Sin } 85^0}{\sin 30°}$$

$$AC = \frac{10 \times 0.99619}{0.5}$$

$$AC = \frac{9.961}{0.5}$$

$$AC = 19.9238$$

$$AC = 19.9 \qquad \text{(3.s.f)}$$

(b) $\frac{BC}{\text{Sin } 65^0} = \frac{10}{\sin 30^o}$

$$BC = \frac{10 \times \text{Sin } 65^0}{\sin 30°}$$

$$BC = 18.126155740733$$

$$BC = 18.1 \qquad \text{(3.s.f)}$$

(c) **Tip:** This can be done using sine or cosine ratios, however we are going to use the sine ratio, so that you can explore the other ways.

$\text{Sin } 65^0 = \frac{h}{10}$ (hint: the perpendicular distance from B to AC)

$$h = 10\text{Sin } 65^0$$

$$h = 9.06307787$$

$$h = 9.06\text{m}$$

(d) The bearing of B from C = 360° – 60° = 300°

Activity (*19.3*) Sine rule with bearing

Questions

1) A girl walks from a shopping centre **X** due west a distance of 8km to a village **Y**. She then changes course and walks on a bearing of 200° to another village **Z** until she is south-west of her starting point. Calculate the distance between X and Z.
2) The bearings of points A and B from a point C are 230° and 120° respectively. Point A is 5m from point B on a bearing of 247°. Find AC.
3) The bearing of P from Q is N40°W. From a point R, 20m due east of Q, the bearing of P is N73°W. Calculate
   a) PQ b) PR
   c) The bearing of Q from P

Answers

1) 17.8
2) 4.25
3) a) 10.7 b) 28.1 c) 140° or S40°E

19.5 THE COSINE RULE

The cosine rule is a formula or a regulation that is used to solve for sides and angles of triangles which are not right-angled. This concept differs from the sine rule in the rules applied.

The Cosine Rule applies where:-

(a) Two sides and the included angle are given, say

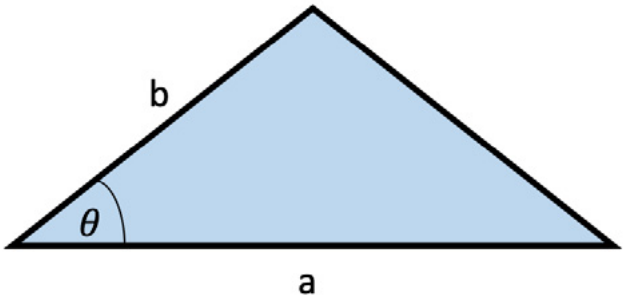


Fig 19.15

(b) all sides are given, say

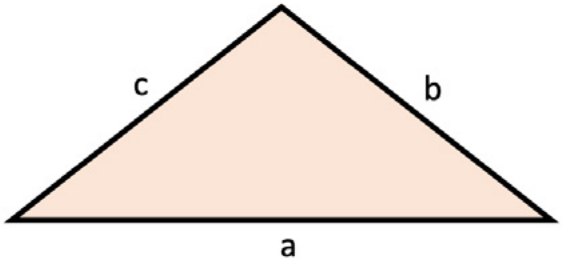


Fig 19.16

Tip: note that this concept is derived from the Pythagorean Theorem and the formulae for sides and angles will be stated without proof.

Given below is a model triangle and the cosine rule formulae for sides and angles.

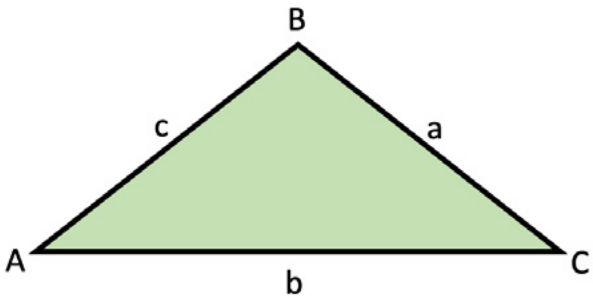


Fig 19.17

Formulae for sides

$$a^2 = b^2 + c^2 - 2bc\text{Cos}A$$
$$b^2 = a^2 + c^2 - 2ac\text{Cos}B$$
$$c^2 = a^2 + b^2 - 2ab\text{Cos}C$$

Formulae for angles

$$\text{Cos}A = \frac{b^2+c^2-a^2}{2bc}$$
$$\text{Cos}B = \frac{a^2+c^2-b^2}{2ac}$$
$$\text{Cos}C = \frac{a^2+b^2-a^2}{2ab}$$

Worked Example [7]

Questions

In triangle ABC below, AB = 6 cm, BC = 9 cm and $A\hat{B}C = 56^o$. Find b

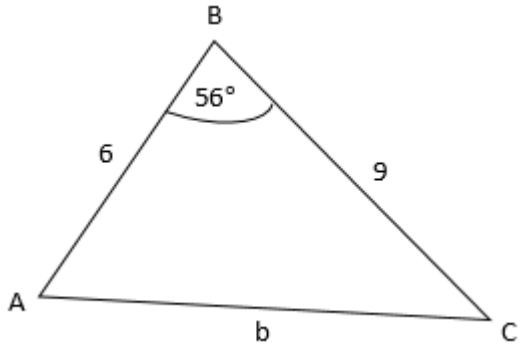


Fig 19.18

Solution

Let us select $b^2 = a^2 + c^2 - 2acCosB$ since we want to find b

$$b^2 = 9^2 + 6^2 - 2 \times 6 \times 9 \times Cos\ 56^o$$

$$= 56.60716643$$

$$= \sqrt{56.60716643}$$

$$= 7.523773417$$

$$= 7.52 \qquad 3.s.f$$

Tip: note the sine rule can now apply to find the remaining angles since "a complete pair" has been found.

Worked Example [8]

Question

Find all the angles in the triangle below.

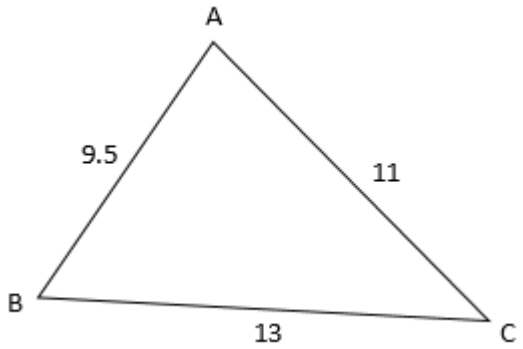


Fig 19.19

Solutions

Finding angle A

$\text{Cos}A = \frac{11^2+9.5^2-13^2}{2\times11\times9.5}$ [workout the denominator and numerator separately]

$\text{Cos}A = \frac{42.25}{209}$

$A = \text{Cos}^{-1}\left(\frac{42.25}{209}\right)$ [use a scientific calculator]

$A = 78.33710464$

$A = 78.3^{o}$

Finding angle B

$\text{Cos}B = \frac{13^2+9.5^2-11^2}{2\times13\times9.5}$ [workout the denominator and numerator separately]

$\text{Cos}B = \frac{138.25}{247}$

$B = \text{Cos}^{-1}\left(\frac{138.25}{247}\right)$ [use a scientific calculator]

$B = 55.96379905$

$B = 56.0^{o}$ (1 d.p)

Finding angle C

$\text{Cos}C = \frac{13^2+11^2-9.5^2}{2\times13\times11}$ [workout the denominator and numerator separately]

$\text{Cos}C = \frac{799}{1144}$

$C = \text{Cos}^{-1}\left(\frac{799}{1144}\right)$ [use a scientific calculator]

$C = 45.69909631$

$C = 45.7^{o}$ (1 d.p)

Check :

$A + B + C = 78.3^0 + 56.0^0 + 45.7^0 = 180^0$

Activity (*19.4*) Cosine rule

Questions

1. Referring to example 7 above, find the lettered sides in the triangles below. All dimensions are given in metres (cm). Also give your answers correct to 3 significant figures.

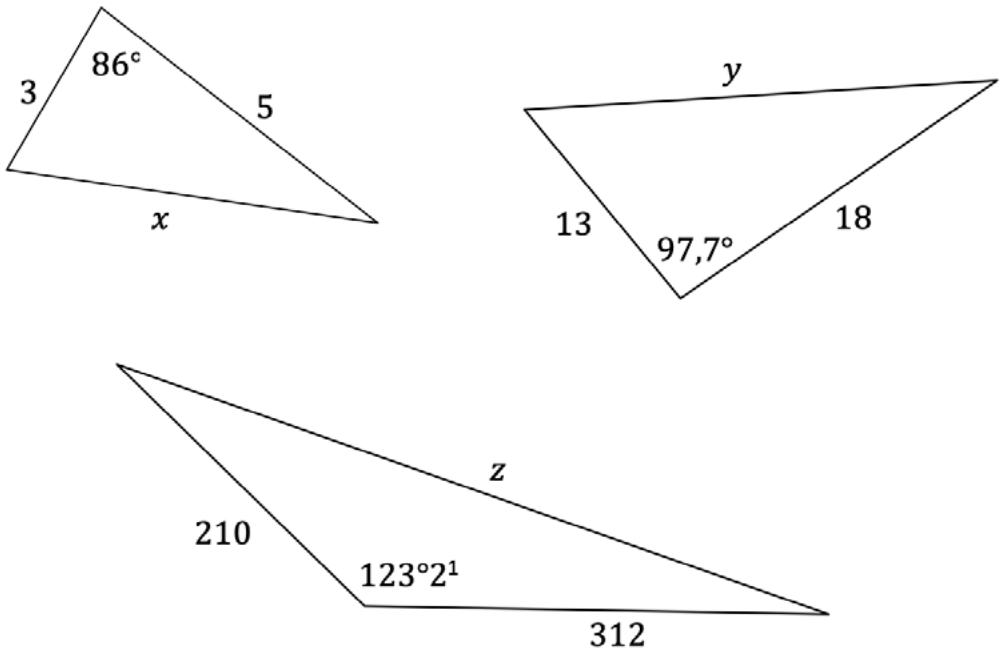


Fig 19.20

2. Referring to example 8 above, find all angles θ in the given triangles. All dimensions are given in metres (mm). Also give your answers correct to 1 decimal place.

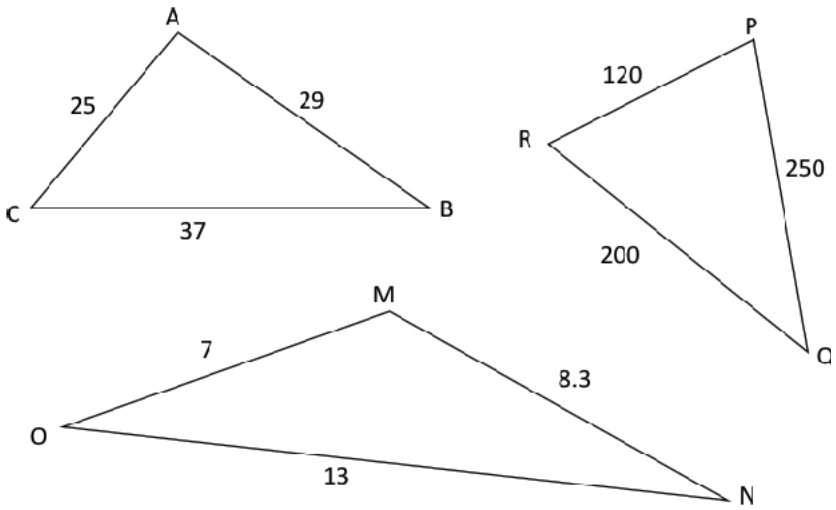


Fig 19.21

Answers

1. $x = 5.65, y = 23.6, z = 461.4$
2. $\widehat{A} = 86.2°$, $\widehat{B} = 42.4°$, $\widehat{C} = 51.4°$

   $\widehat{P} = 52.0°$, $\widehat{Q} = 22.2°$, $\widehat{R} = 99.7°$

   $\widehat{M} = 116.1°$, $\widehat{N} = 28.9°$, $\widehat{O} = 35.0°$

19.6 COSINE RULE WITH BEARING

Now that you able to solve for sides and angles using the Cosine rule, let's move a step further into the application of the same concept on bearing.

Tip: for each bearing question you should understand the diagrammatical representation of the question so that you can use the calculation learnt in the examples discussed earlier.

Worked Example [8]

Questions

Three boreholes in a village are situated as follows. Borehole A is 4km due east of borehole B. Borehole C is 3.5km on a bearing of 116° from borehole B. How far is it from borehole C to borehole A. Find the bearing of borehole C from borehole A

Solutions

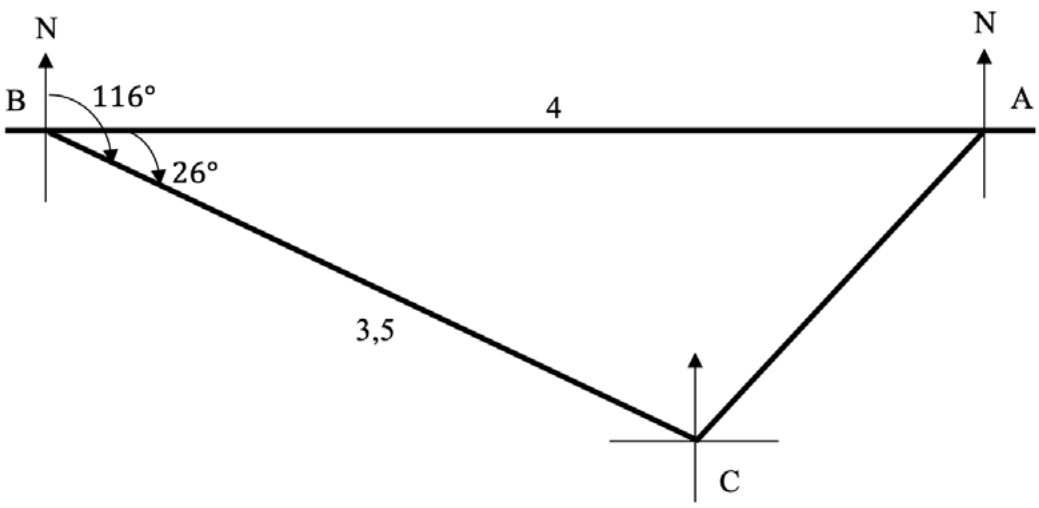


Fig 19.22

$b^2 = 3.5^2 + 4^2 - 2 \times 3.5 \times 4 \times \text{Cos } 26^o$

$b^2 = 3.084$ (4 s.f)

$b = \sqrt{3.084}$

$b = 1.756$

$b = 1.76$ (3 s.f)[that's the distance between borehole A and C]

To find the bearing of C from A we need to find angle A. The working is shown below.

$\text{Cos}A = \frac{1.76^2+4^2-3.5^2}{2\times1.76\times4}$ workout the denominator and numerator separately

$\text{Cos}A = 0.48633523$

$A = \text{Cos}^{-1}(0.4863)$ (4.s.f) use a scientific calculator

$A = 60.90°$

$A = 60.9^o$ 3.s.f

Therefore, the bearing of C from A is 360° – (90° + 60.9°) = 209.1° = 209° three-figure bearing.

Activity (*19.5*) Cosine rule with bearing

Questions

In each question come up with a sketch diagrams to represent the given information

1) A train travels 10km on a bearing of 040° and then 15km on a bearing of 065°
   (a) How far is the train from its starting point
   (b) Calculate the bearing of the train from its starting
2) If you walk 100m on a bearing of 025° and 400m due east. How far are you from your starting point?
3) From the door of a classroom, Chipo is 5m away on a bearing of S40°E and Zenzo is 3m away on a bearing of S70°W. Find the distance between Chipo and Zenzo

Answers

1)a) 24.4 b) 50°

2) 451 3) 6.65

REFLECTION

- The sine rule applies where there is a complete pair of a side and an angle given.
- note that the Sine and Cosine Rule concepts are derived from the Pythagorean Theorem and the formulae for sides and angles can be stated without proof.
- For each bearing question you should understand the diagrammatical representation of the question

19.7 Summary

From this unit you have learnt how to find the area of a triangle using $A = \frac{1}{2}ab\,\mathrm{Sin}\theta$, where the triangle has no height. You have also learnt how to solve for sides and angles using the sine and cosine rules. The application of sine and cosine was also done through the use of bearing and a few questions on angles of elevation and depression.

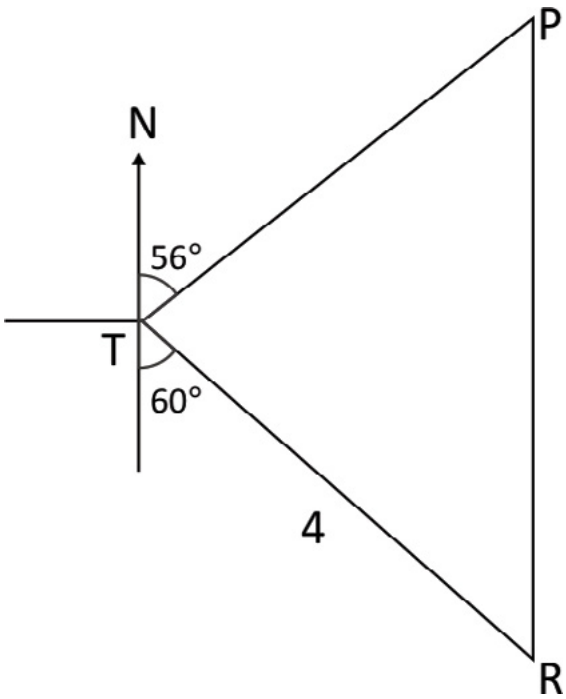
19.8 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

19.9 Assessment Test

1. Find the area of triangle ABC correct to 3 significant figures if
   a) AB = 9cm, BC = 12cm and $\widehat{B} = 40°$
   b) AB = 20 cm, AC = 13 cm and $\widehat{A} = 115°$
2. In triangle PQR $\widehat{P} = 83°$, p = 112mm, r = 147mm, Calculate $\widehat{R}$
3. In triangle XYZ,$\widehat{Y} = 31°39^{I}$, $\widehat{Z} = 55°42^{I}$ and $x = 23.1m$. Solve the triangle completely
4. Calculate the length of the opposite side of the given angle in each of the triangles XYZ. Give your answer correct to 3sf
   a) $\widehat{X} = 127°$, y = 9cm and z =15 cm
   b) $\widehat{Y} = 61.7°$, x = 3m and z = 6m
   c) $\widehat{Z} = 136°54^{I}$, x = 4.5m and y = 6.7m
5. Calculate the angles of the triangles ABC whose sides are given in mm. Give the final answers to the nearest 0.1°
   a) a = 10, b = 13 and c = 17
   b) a = 25, b = 19.5 and c = 29
   c) a = 2, b = 4 and c = 5

6.



The diagram shows three points, P, R and T which are on level ground and TR = 4 km. From T, the bearing of P is N 56° E, the bearing of R is S 60°E and P is due north of R.

(i) Calculate

1. the shortest distance between line PR and T,

2. PR.

(ii) From an aeroplane flying directly above point T the angle of depression of R is 20,3°.

Calculate the height of the aeroplane above the ground at T.

4028/02 N2013 Question 10(b)

UNIT 20 - TRAVEL GRAPHS

CONTENTS

20.1 INTRODUCTION

In this unit we are going to look on travel graphs which involves, speed-time graph, velocity –time and distance–time graphs. We are going to interpret them, draw them and solve questions related to each type of graph. In drawings we are going to exercise drawing both curved and non-curved graph for each type of a travel graph. We hope that you are going to enjoy the unit.

OBJECTIVES

After going through this unit, you should be able to:

- calculate speed using the distance – time graph.
- interpret the speed-time, velocity–time and the distance graphs.
- calculate acceleration and distance from the speed/velocity time graph.
- draw both curved and non-curved velocity–time graphs.

KEY TERMS

Distance- any length covered by a moving object.

Displacement – distance or length covered by a moving object in a specific direction.

Velocity – is the displacement per unit time.

Acceleration- is the rate of change of velocity or speed with time.

Deceleration-is the decrease in velocity/speed with time.

TIME :- You are advised to spend not more than 10 hours in this unit.

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems on Travel Graphs as possible for you to grasp all the concepts in this topic. Revisit the topic on gradients of straight lines and quadratic graphs before you begin this topic.

20.2 SPEED-TIME GRAPHS

The diagram shows a speed–time graph of an object in 3 stages which are OQ, QP and PR

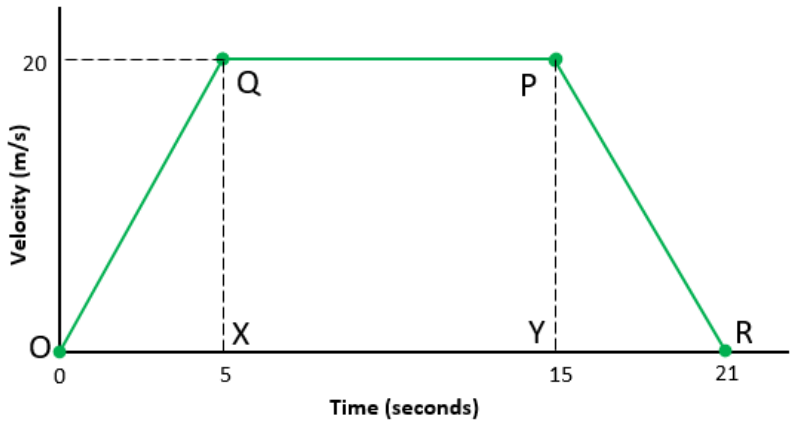


Fig 20.1

1. Stage OQ

   During this stage the object starts from rest, that is 0 m/s and it accelerates uniformly up to 20 m/s

   Acceleration at this stage is found by calculating the gradient of the line OQ

   $$\text{Acceleration} = \frac{\text{change in speed}}{\text{change in time}}$$

   $$= \frac{20-0}{5-0}$$

   $$= 4\text{m/s per second or } 4\text{m/s}^2$$

   We can also find the distance travelled by an object at this stage.

Distance travelled = area under the graph

= area of triangle OQX

$= \frac{1}{2}bh$

$= \frac{20\times5}{2}$

= 50m

2. Stage QP

During this stage the object is moving at a constant speed of 20m/s for 10 seconds

The acceleration at this stage is zero

We can also calculate the distance travelled by an object at this stage

Distance = area under the graph

= area of a rectangle XQPY

= L×W

= 20×10

= 200m

3. Stage PR

During this stage the object is now slowing down its speed uniformly until rest.

If an object is reducing its speed it means it is decelerating or retarding.

The acceleration at this stage is negative

Acceleration $= \frac{\text{change in speed}}{\text{change in time}}$

$= \frac{0-20}{21-15}$

$= -3\frac{1}{3}$ m/s per second

Therefore it means that the deceleration of the object is $3\frac{1}{3}$m/s

Finding the distance travelled by an object at this stage we calculate area under the graph

Distance = area of a triangle YPR

$= \frac{1}{2}bh$

$= \frac{20\times6}{2}$

= 60m

Note it: we can calculate distance for the whole journey and the average speed for the whole journey

i. Total distance travelled

= 50m + 200m + 60m

= 310m

ii. Average speed for the whole journey

$= \frac{\text{total distance travelled}}{\text{total time taken}}$

$= \frac{310}{21}$

$= 14\frac{16}{21}$ m/s

Worked Example [1]

Questions

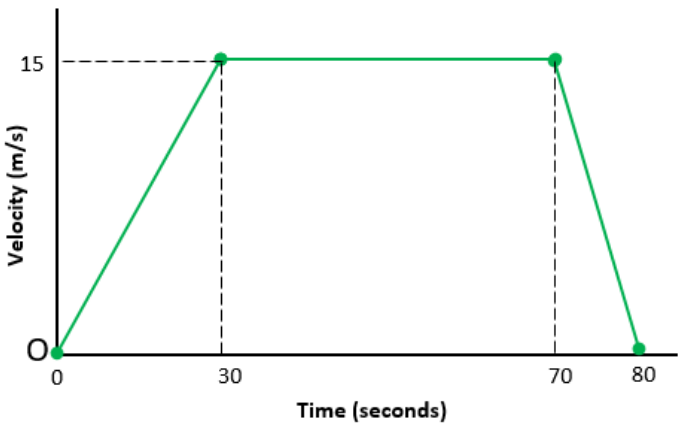


Fig 20.2

The diagram above shows a speed-time graph of a train

a) Calculate the acceleration of the train during the first 30 seconds

b) Calculate the distance the train travels from rest before it begins to decelerates

c) Given that the total distance travelled for the whole journey is 900m find the total time taken for the whole journey.

d) Calculate the deceleration of the train as it comes to rest.

Solutions

a) Acceleration $= \frac{\text{change in velocity}}{\text{change in time}}$

$= \frac{15-0}{30-0}$

$= 0{,}5 m/s^2$

b) Distance travelled = area under the graph

= area of a trapezium

$= \frac{1}{2}(a + b)h$

$= \frac{(70+40)15}{2}$

= 825m

c) The total distance travelled is 900m, but the distance travelled up to 70 seconds is 825m

The distance travelled by the train in the last seconds at deceleration stage

= 900m − 825m

= 75m

So in order to find the value of T we have to consider the area of the graph

area of the graph

$= \frac{1}{2}bh$

$75 = \frac{(T-70)15}{2}$

150 = 15T – 1050

1200 = 15T

$\therefore T = 80 \text{ seconds.}$

d) Acceleration $= \frac{\text{change in velocity}}{\text{change in time}}$

$= \frac{15-0}{70-80}$

$= -1{,}5$

$\therefore$ the decelaration is $1{,}5\ \text{m/s}^2$

Worked Example [2]

Questions

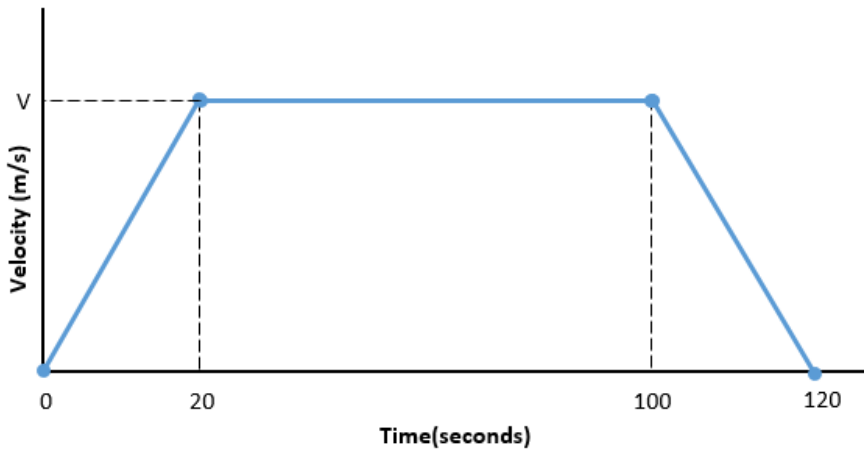


Fig 20.3

The diagram above shows a speed time graph of a commuter omnibus travelling from Mbare Musika to town. Given that the total distance travelled is 600m. Calculate

a) the maximum speed V m/s
b) the acceleration of the commuter omnibus during the first 20 seconds
c) the distance travelled in the first 30 seconds
d) the average speed for the whole journey

Solutions

a) In order to find the value of V we have to make use of the area under the graph since we are given the total distance travelled.

Distance = area under the graph

600 = Area of a trapezium

$600 = \frac{1}{2}(a + b)V$

1200 = (120 + 80)V

1200 = 200V, dividing by 200 both sides we get;

V = 6m/s

$\therefore$ the maximum speed is 6m/s

b) Acceleration = $\frac{\text{change in velocity}}{\text{change in time}}$

$= \frac{6-0}{20-0}$

= 0,3 m/s per second

c) Distance travelled = Area under the graph

=area of a trapezium

$= \frac{1}{2}(a + b)h$

$= \frac{(30+10)6}{2}$

=120m

d) Average speed = $\frac{\text{total distance travelled}}{\text{total time taken}}$

$= \frac{600}{120}$

= 5 m/s

Worked Example [3]

Questions

The diagram below shows the motion of an object travelling at 24m/s for 8 seconds and then slows down at a constant speed for a further 4 seconds until it rests.

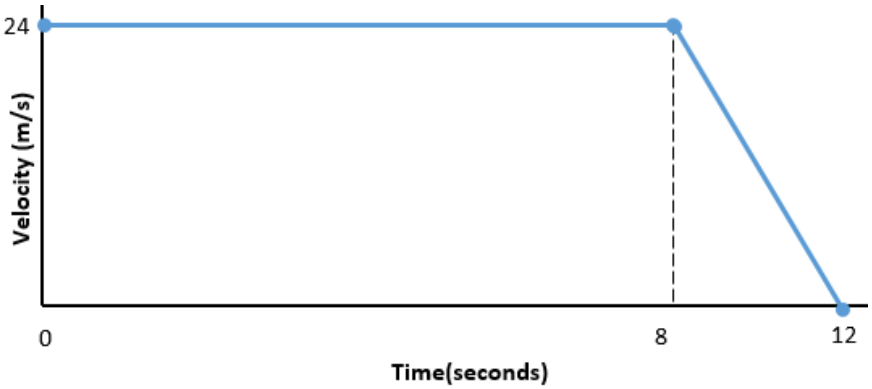


Fig 20.4

Calculate

a) the distance covered in the 8 seconds
b) the speed of the object after 10 seconds
c) the average speed of the object during 12 seconds

Solutions

a) Distance = Area under the graph

= area of a rectangle

$= L \times W$

$= 24 \times 8$

$= 192m$

b) First we have to calculate the acceleration of the object during the last 4 seconds, this is where the 10th second lies

Acceleration $= \frac{\text{change in velocity}}{\text{change in time}}$

$= \frac{0-24}{12-8}$

$= \frac{-24}{4}$

$= -6m/s^2$

Since we have calculated the acceleration we use it to find the speed in the 10th second

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

$-6 = \frac{24-v}{8-10}$(where v represents the speed of an object at the 10th second)

$-6 = \frac{24-v}{-2}$, multiplying both sides by the denominator we get;

$12 = 24 - v,$

$\therefore v = 12m/s.$

20.3 VELOCITY TIME GRAPH

Note it: the velocity-time graph is the same as the speed-time graph. The difference is that in the velocity-time graph we are considering displacement (distance in a specific direction of an object) while in the speed-time graph we are considering distance of an object without having a specific direction.

Worked Example [4]

Questions

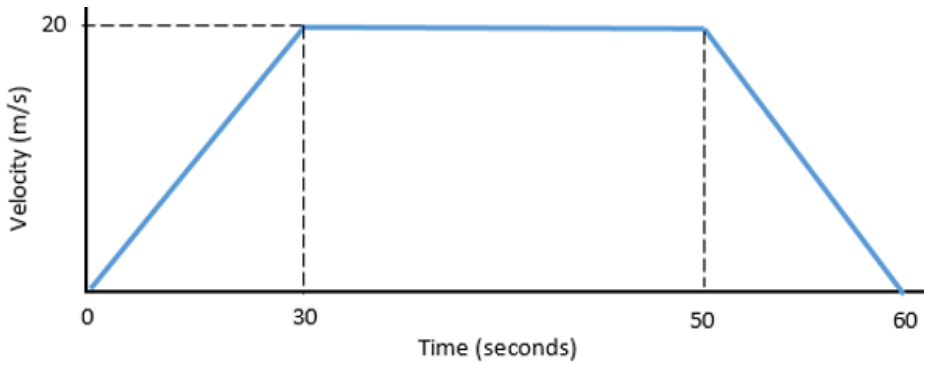


***Fig 20.5** above shows the velocity-time graph for a particular journey.*

Calculate

a) the distance travelled in the first 30 seconds
b) the velocity when the time is 40 seconds
c) the deceleration during the last 10seconds

Solutions

a) Distance = area under the graph

= area of a triangle

$= \frac{1}{2}\mathrm{bh}$

$= \frac{30 \times 20}{2}$

= 300m

b) At the 30th second the velocity reached to a maximum of 20 m/s and maintained up to the 50th second

So the velocity when the time is 40 seconds = 20 m/s

Acceleration $= \frac{\text{change in velocity}}{\text{change in time}}$

$= \frac{20-0}{50-60}$

$= -2\mathrm{m/s^2}$

$\therefore$ the deceleration $= 2\mathrm{m/s^2}$

Remember (deceleration has no negative sign)

Worked Example [5]

Questions

The diagram below is a travel graph showing the motion of an object which has a starting velocity of 25m/s. The object decelerates at $4\mathrm{m/s^2}$ in 2 seconds, travels at a constant speed for the next 8 seconds and finally accelerates for 2 seconds until it reaches a velocity of 30m/s.

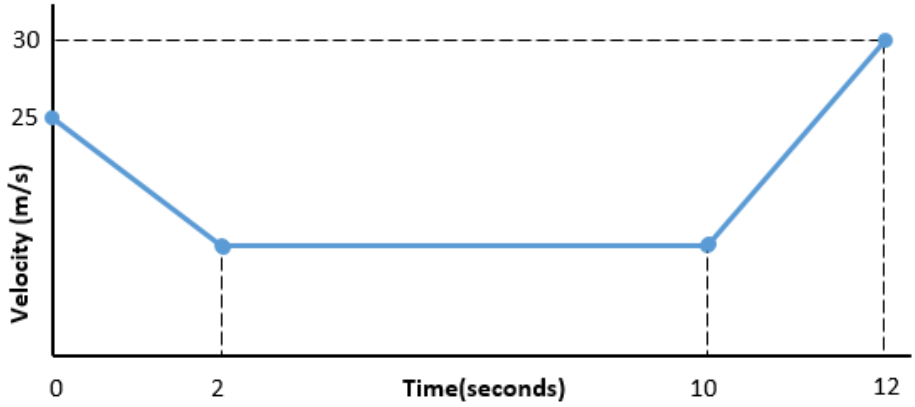


Fig 20.6

Find

a) its speed after 6 seconds
b) its speed after 11seconds
c) distance of the object for the whole journey
d) the average velocity for the whole journey

Solutions

a) First we have to find the velocity of an object at the 5th second since we have the deceleration of $4m/s^2$

Acceleration $= \frac{\text{channge in velocity}}{\text{change in time}}$

$-4 = \frac{25-v}{0-2}$

$-4\,(-2) = 25 - v$

$8 = 25 - v$

$\therefore v = 17m/s$

After 2 seconds its speed had dropped to 17m/s meaning that it maintained this speed up to the 10th second

Therefore the speed of the object after 6 seconds is 17m/s

b) From the 10th second up to the 12th second the object accelerates at a uniform velocity

Acceleration = $\frac{\text{channge in velocity}}{\text{change in time}}$

$=\frac{30-17}{12-10}$

$=\frac{13}{2}$

6,5 m/s^2

Since the acceleration is 6,5m/s^2, we make use of that to find the velocity at the 11th second

Acceleration = $\frac{\text{change in velocity}}{\text{change in time}}$

$$6{,}5 = \frac{30 - v}{12 - 11}$$

$$6{,}5 = 30 - v$$

$$\therefore v = 23{,}5\text{m/s}$$

So the velocity of an object after 11 seconds was 23,5 m/s

c) Distance = area under the graph

= (area of trapezium1) + (area of a rectangle) + (area of trapezium2)

$= \frac{1}{2}(a + b)h$ +(L×W) +$\frac{1}{2}(a + b)h$

$= \frac{(25+17)2}{2} + (17 \times 8) + \frac{(17+30)2}{2}$

$= 42 + 136 + 47$

$= 225\text{m/s}.$

d) Average velocity

$= \frac{\text{total distance travelled}}{\text{total time taken}}$

$= \frac{225}{12},$

$= 18\frac{3}{4}\text{m/s}$

Worked Example [6]

Questions

The diagram below shows a velocity time graph of a car journey in 150 seconds.

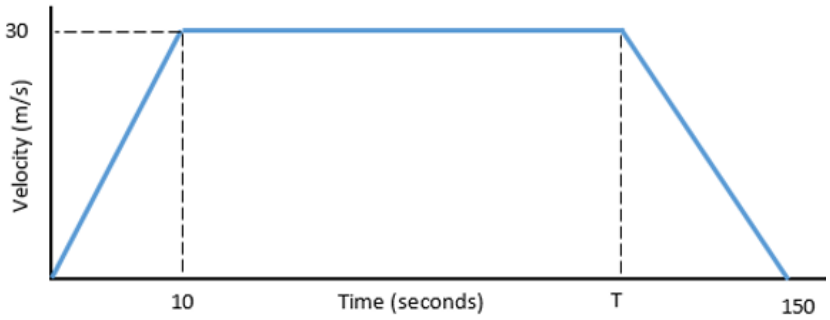


Fig. 20.7

a) Calculate the acceleration of the car during the first 10 seconds

b) Given that the final deceleration is 0,4 m/s^2, calculate

   i. the value of T

   ii. the total distance travelled by the car in km.

Solutions

a) Acceleration $= \frac{\text{change in velocity}}{\text{change in time}}$

$$= \frac{30-0}{10-0}$$

$$= 3\text{m/s}^2.$$

b) If the deceleration is 0,4 it means it's a negative acceleration of 0,4

   i. Calculating the value of T

   $-0{,}4 = \frac{30-0}{T-150}$, [clearing the denominator we get];

   $-0{,}4(T-150) = 30$, [removing brackets we get];

   $-0{,}4T + 60 = 30$, [collecting like terms we get];

   $-0{,}4T = -30$, [dividing throughout by $-0{,}4$]

   $\therefore T = 75.$

ii. Total distance travelled =area under the graph

=area of a trapezium

$=\frac{1}{2}(a+b)h.$

$=\frac{(150+65)30}{2}$

=6450m

=6,45km

Worked Example [7]

Questions

In this example we are going to consider how to draw curved velocity-time graph and answer the questions that follow.

The velocity (v m/s) of an object at time (t seconds) is given by $\mathbf{v} = (1-t)(t-4)$

t	0	1	2	3	4	4,5	5	6
v	−4	**p**	2	2	0	**q**	−4	−10

a) Calculate the value of p and the value of q

b) Using a scale of 2 to represent 1 second on the horizontal axis and 2cm to represent 2m/s on the vertical axis, draw the graph of v = (1−t) (t−4) for $0 \leq t \leq 6$ and $-10 \leq v \leq 4$

c) Use the graph to estimate ,

(i) the velocity when the acceleration is zero.

(ii) the time when the acceleration is zero.

(iii) the acceleration when $t = 5$.

(iv) the distance covered by the object from t=1 and $t = 4$.

Solutions

a)

$$p = (1 - 1)\,(1 - 3)$$
$$p = 0$$

finding the value of q

$v = (1-t)\,(t-4)$

$q = (1-4,\ 5)\,(4,\ 5-4)$

$q = -3{,}5\ (0{,}5)$

$q = -1{,}75$

b) graph

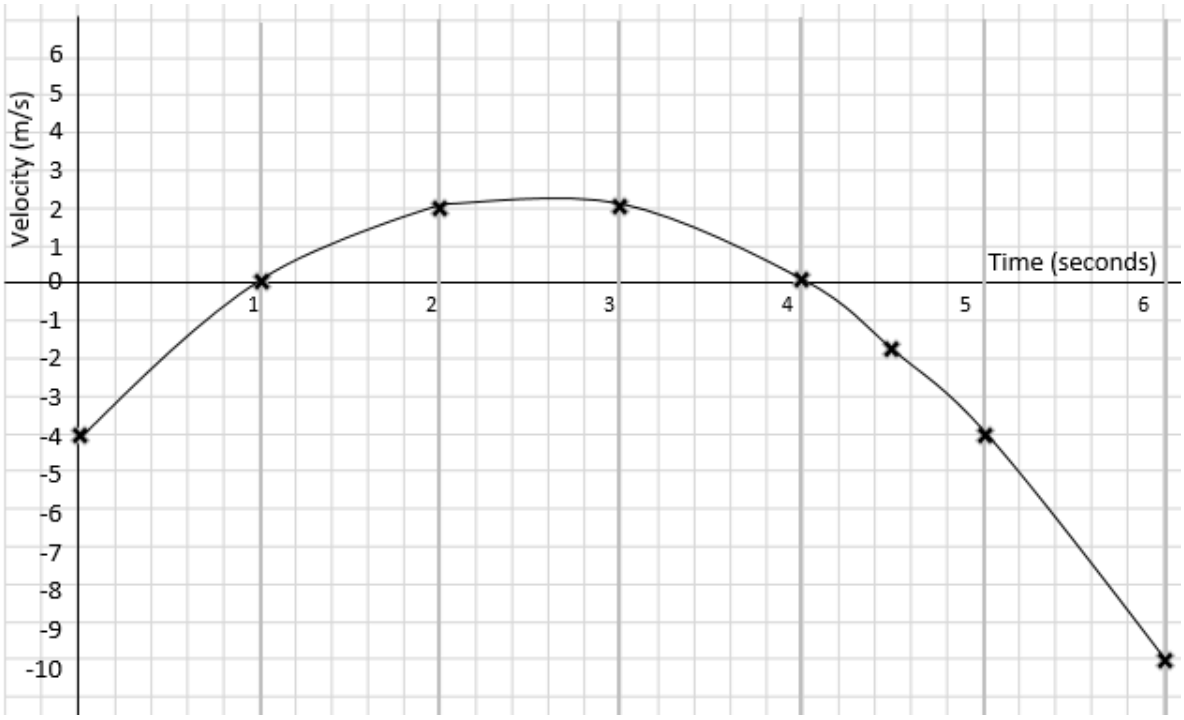


Fig 20.8

c)

(i) Velocity when acceleration is 0 is at the turning point of the graph, so the velocity is 2,2m/s

(ii) The time when the acceleration is 0 is at the turning point of the graph again so, the time is 2,5 seconds

(iii) The acceleration when time is 5 (at this point we have the gradient at $t = 5$)

by drawing the tangent at $t = 5$ we get

Acc $= \frac{\Delta v}{\Delta t}$ [meaning change in velocity divided by change in time]

Acc $= -\frac{2}{0,5}$

$\therefore$acc $= -4$

(iv) By finding the distance travelled we have to calculate the area under the curve from $t = 1$ up to $t = 4$(the shaded area)

We can estimate the area under curve by method of counting squares or dividing the shaded area into workable shapes

In this case a trapezium can be used to estimate the area

Area $= \frac{1}{2}(a + b)h$

Area $\approx \frac{1}{2}(1 + 3) \times 2$

Area $\approx 4\text{units}^2$

$\therefore$ the distance covered by an object $\approx 4\text{m}$

20.4 DISTANCE TIME GRAPHS

The gradient of the distance-time graph gives us speed or velocity. Let us consider the following examples

Worked Example [8]

Questions

A stone is thrown into the air. Its height h meters after t seconds is given by the formula

$h = 60 + 30t - 5t^2$

Time(seconds)	0	1	2	3	4	5	6	7	8
Height(metres)	60	85	100	**p**	100	85	60	**q**	-20

Find the value of **p** and **q**

a) using the scale of 2cm:1 second on the horizontal axis and 2cm:20m on the vertical axis draw the graph $h=60+30t-5t^2$

b) find the maximum height reached by the stone.

c) find the velocity when t=2seconds.

d) find the times reached by the stone when the height was 90m.

Solutions

In order to find the value of p and q we have to substitute 3 and 7 in the equation respectively

$\mathbf{p} = 60 + 30t - 5t^2$

$\mathbf{p} = 60 + 30\,(3) - 5(3^2)$

$\mathbf{p} = 60 + 90 - 45$

$\therefore \mathbf{p} = 105$

$\mathbf{q} = 60 + 30t - 5t^2$

$\mathbf{q} = 60 + 30 - 5(7^2)$

$\mathbf{q} = 90 -$

$\therefore \mathbf{q} = 25$

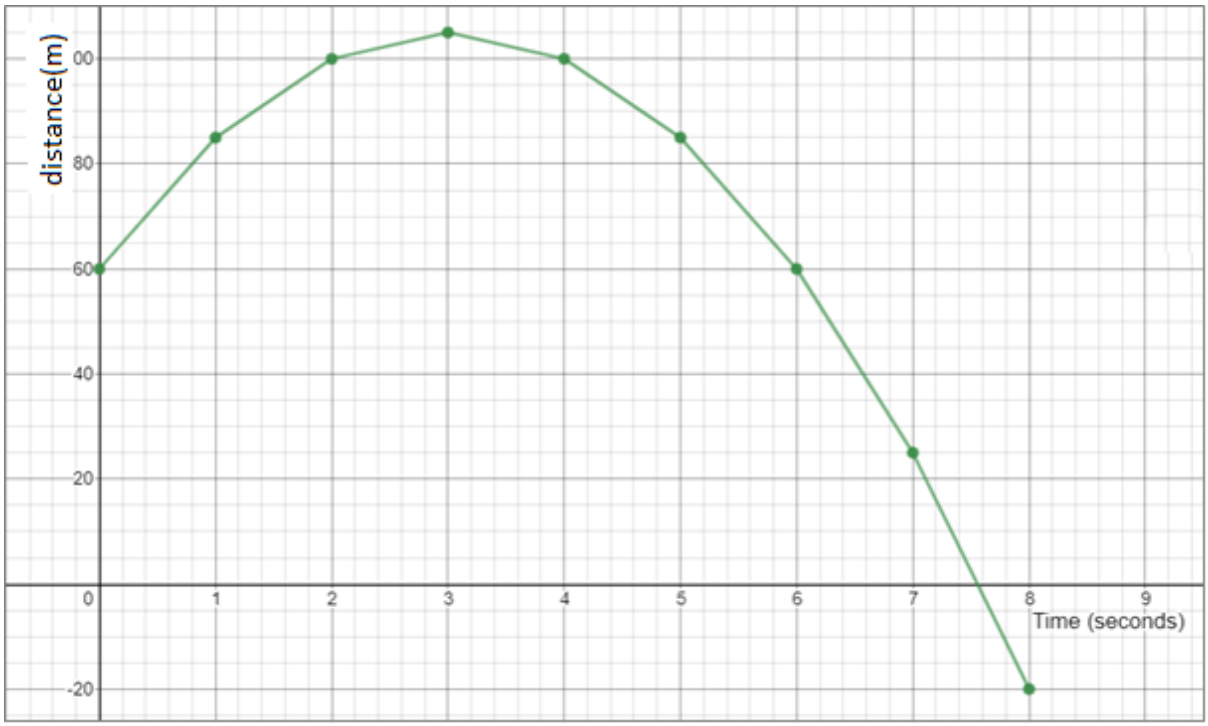


Fig 20.9

a) maximum height is 104m(this is the turning point of the graph)
b) velocity at t=2 seconds is 100m
c) at 80m time is 0,8 seconds and 5,2 seconds

Activity (*20.1*) Distance time graphs

Questions

1) The diagram below shows a speed-time graph of the last 10seconds of a car journey. It travels at a constant speed of 62m/s for 4 seconds and slows down uniformly, first to 32m/s then to rest after a further 5 seconds and 1 second respectively

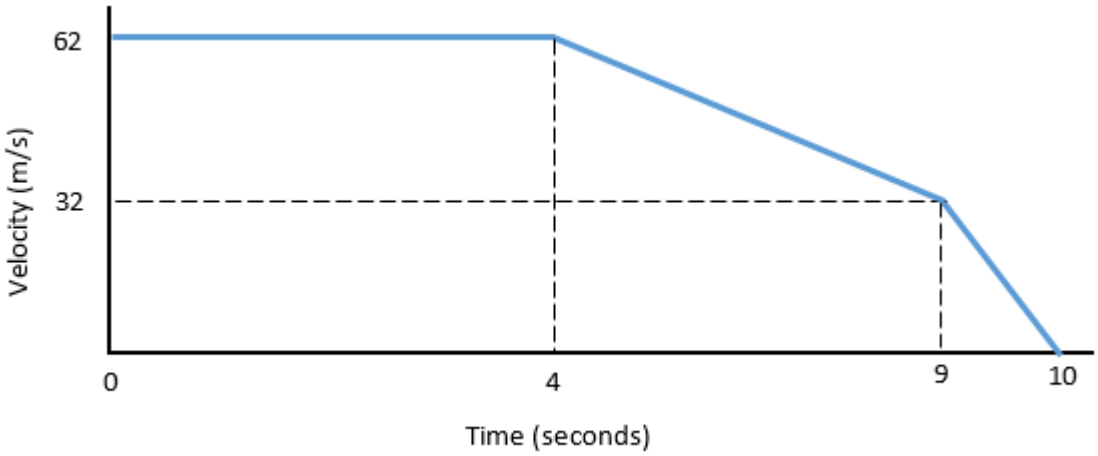


Fig 20.10

Calculate

a) the speed of the car 2 seconds before it stopped
b) the distance travelled by the car before it starts to decelerates
c) the average speed of the car during the whole 10 seconds.

2) The diagram below is a velocity-time graph of the motion of an object decelerating uniformly from 60m/s until it reaches the velocity of 10m/s. The object then moved on with a constant velocity of 10m/s and then retards until rest. Given that the total distance travelled is 1,5km

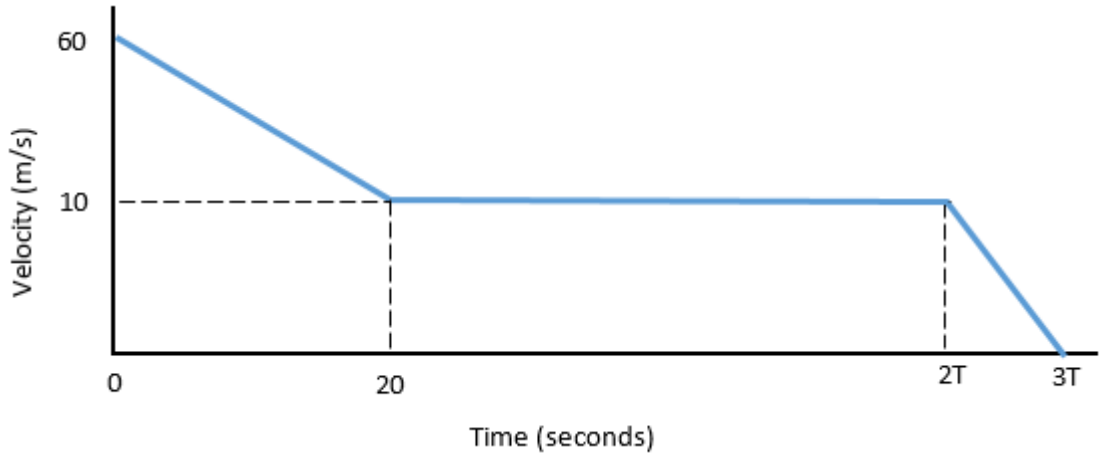


Fig 20.11

Find

a. the deceleration during the first 10 seconds.
b. the value of T.
c. the average velocity for the whole journey.

Answers

1. (a) 38	(b) 248	(c) 499
2. (a) 2.5	(b) 40	(c) 12,5

20.5 Summary

This unit covered Travel graphs. In this unit we have noted that, the gradient of the velocity-time or speed-time graph gives us the acceleration the gradient of the distance–time graph gives us speed or velocity, the area under a velocity-time or speed-time graph gives us distance, a negative acceleration gives us deceleration and that deceleration is also called retardation.

20.6 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK

5. Rayner D (2005) Extended Mathematics. Oxford New York

20.7 Assessment Test

1).

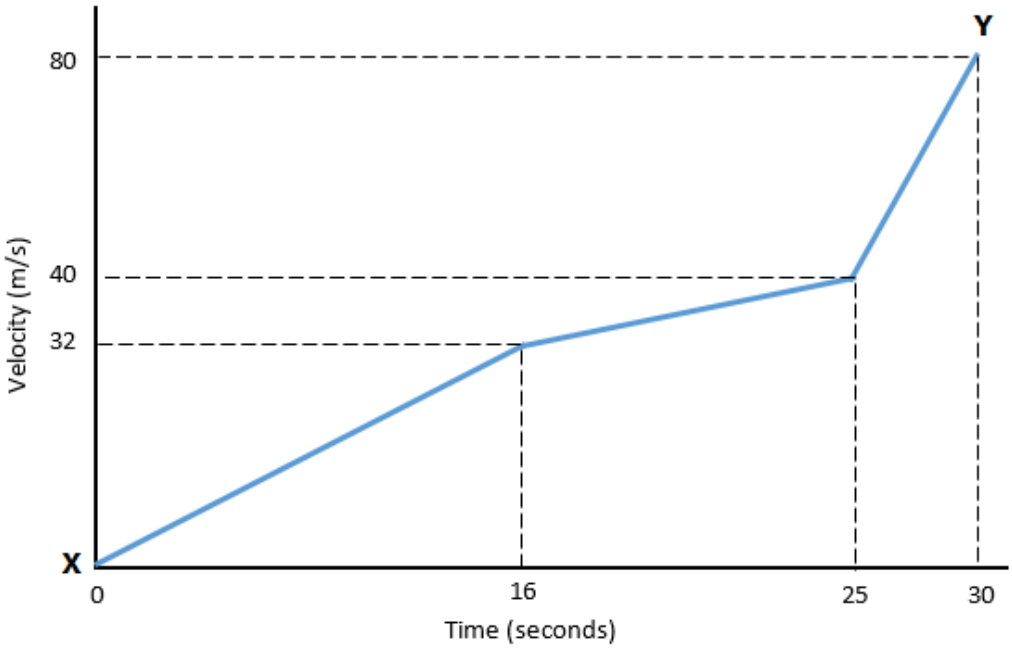


Fig 20.12

Find

a) the time it takes for the object to reach the velocity of 20m/s.
b) the value of time (t) for which speed (v)=35m/s.
c) the distance travelled in the first 18 seconds.
d) the average velocity for the whole journey.

3. The diagram is the velocity–time graph of an object which decelerates uniformly from a velocity of 90m/s to a velocity of 60m/s in 10 seconds. It then decelerates uniformly to rest in a further 5 seconds.

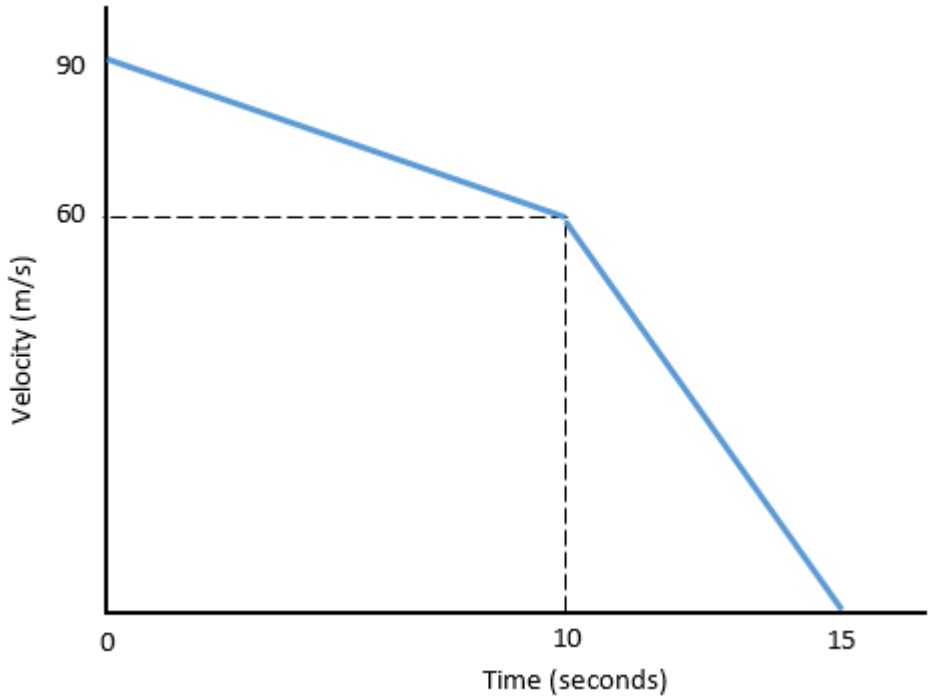


Fig 20.13

Calculate the

a) total distance by the object during the 15seconds…….[2]
b) the average velocity of the object during 15 seconds…..[2]
c) deceleration of the object during the last 5 seconds…..[2]

4. The following is an incomplete table of values of a velocity-time graph of an object in motion given by the equation $v = 12 + 5t - 2t^2$

Time (t seconds)	0	1	2	3	4	5
Velocity(m/s)	12	15	Q	9	0	-13

a) Find the value of q
b) using a suitable scale on both axes draw the graph $v = 12 + 5t - 2t^2$
c) use the graph to
   (i) find the times when the velocity is 13m/s
   (ii) the acceleration of an object when time is 3 seconds
   (iii) the distance travelled $t = 1\text{second to } t = 3\text{seconds}$

UNIT 21 – PROBABILITY

CONTENTS

21.1 INTRODUCTION

You are invited to a party where they are serving just two types of drinks, Coke and Fanta. There are more Fanta drinks as compared to Coke but there is an overwhelming number of party goers who want to take Coke. A method needs to be devised to issue out the most demanded drink. A card system is designed with some labelled Coke and some Fanta. The name of the drink labelled on the card you pick will determine the type of drink you will be served. This scenario of choosing drinks brings us yet to a very interesting topic of Probability.

OBJECTIVES

After going through this unit you should be able to

- define the term probability
- describe both experimental and theoretical probability
- solve problems involving both experimental and theoretical probability.
- use probability rules to compile probability of single and combined events.

KEY TERMS

Experiment: – It is any activity with an observable result, that is, a well-defined set of possible outcomes.

Outcome: – It is a possible result of an experiment

Sample space: – It is a set of all possible outcomes of an experiment. The sample space for the experiment of flipping a coin is heads and tails.

Possible outcomes: – These are the possible results of an experiment

Probability: - is the study of chance or the likelihood of an event happening.

Required outcomes: – that is what has actually happened.

Event: - a set of outcomes of an experiment to which a probability is assigned, that is, it is the result of an experiment

Random event: - A random event is something unpredictable hence you can never give it an exact value or probability.

TIME: You should not spend more than 10 hours in unit.

STUDY SKILLS

The key to understanding probability is mastering the two basic rules of addition and multiplication associated with probability. Some probability experiments are easily understood by practically carrying them out yourself. Practice is the top most important skill required in understanding probability.

21.2 PROBABILITY DEFINITION

Probability is the study of chance or the likelihood of an event happening. Probability is also known as the measure of the likelihood of something happening. Usually this measure is given as a fraction that lies between 0 and 1 [$0 \leq P(A) \leq 1$]. If probability is 0 it means that event will not happen. If on the other hand probability is 1 it means that event is certain and it will certainly happen.

21.3 EXPERIMENTAL PROBABILITY

Using a coin or bottle top throw it 20 times and record the outcomes. One side of the coin is the head (H) and the other tail (T). Out of the 20 throws, how many are heads and how many are tails? Let us say you obtained 11 heads and 9 tails, the probability of getting a head is $\frac{11}{20}$ and the probability of a tail is $\frac{9}{20}$. These results are determined by the experiment you would have undertaken. Experimental probability is based on observations or past experiences. It is applicable when betting.

If a six-sided die is rolled, the following numbers are likely to appear (1; 2; 3; 4; 5; 6). These 6 numbers comprise all the possible outcomes on a six-sided die.

When a die is rolled and a 6 appears, the appearance of a 6 becomes the required outcome.

In a set of playing cards there are 52 cards distributed as Diamonds, Hearts, Spades and Clubs.13 from each suit. Red cards are Diamonds and Hearts. Black cards are Spades and Clubs. If a card is chosen at random it is chosen from the 4 designs of the cards. A complete set of cards contains 13 different types of cards. These are 2;3;4;5;6;7;8;9;10, J(Jack),Q (Queen),K (King) and A (Ace)

21.4 THEORETICAL PROBABILITY

Theoretical probability is a method to express the likelihood that something will occur. It is calculated by dividing the number of favourable (required) outcomes by the total possible outcomes.

Theoretical probability = $\frac{\text{number of required outcome}}{\text{number of possible outcomes}}$

Worked Example [1]

Questions

Get 10 cards of the same size and write the following letters

A; A ;A; A ;A; B; B; B ;B; C. Put the 10 cards in an envelope. Take a card from that envelope at random. From the 10 cards you may pick an A, B or C.

Find the probability of getting

a) A b) B c) C d) D

Solutions

Formula for calculating probability $= \frac{\textbf{number of required outcomes}}{\textbf{number of possible outcomes}}$

Probability of getting A $= \frac{5}{10} = \frac{1}{2}$

Probability of getting B $= \frac{4}{10} = \frac{2}{5}$

Probability of getting C $= \frac{1}{`10}$

Probability of getting D = 0

Tip: Probability of getting A can be written as P (A) $= \frac{1}{2}$

Let us look again at another example.

Worked Example [2]

Questions

A bag contains 7 Oranges, 8 Apples and 5 Mangoes. A fruit is taken from the bag at random. What is the probability that the fruit taken is

a) Orange b) Apple

c) Mango d) Guava

e) either Orange or Apple f) neither Orange nor Apple

Solutions

(a) P(Orange)$=\frac{7}{20}$

(b) P(Apple) $= \frac{8}{20} = \frac{2}{5}$

(c) P(Mango) $= \frac{5}{20} = \frac{1}{4}$

(d) P(Guava) = 0

(e) P(Orange or Apple) $= \frac{15}{20} = \frac{3}{4}$

(f) P(Orange nor Apple) $= \frac{5}{20} = \frac{1}{4}$

Now that we have gone through a few examples, attempt the following activity.

Activity (*21.1*) Theoretical probability

Questions

1) A crate of drinks contains 9 Coke, 10 Fanta and 5 Sprite. You are asked to choose a bottle at random from the crate. What is the probability of choosing?

   a) Coke
   b) Fanta
   c) Sprite
   d) Lemon
   e) Either Coke or Sprite
   f) Neither Coke nor Fanta

2) A six sided die is rolled once. What is the probability of getting a

   a) 5
   b) 5 or 6
   c) Prime number
   d) Perfect square
   e) Even number
   f) 10

3) A farmer has 30 goats and 10 sheep. He would like to slaughter one of these animals on a field day. What is the probability of slaughtering

   a) a goat
   b) a sheep
   c) a pig

Answers

1). (a) $\frac{3}{8}$ (b) $\frac{5}{12}$ (c) $\frac{5}{24}$ (d) 0 (e) $\frac{7}{12}$ (f) $\frac{5}{24}$

2). (a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$ (e) $\frac{1}{2}$ (f) 0

3). (a) $\frac{3}{4}$ (b) $\frac{1}{4}$ (c) 0

21.5 MUTUALLY EXCLUSIVE EVENTS

Worked Example [3]

Questions

A box contains 5 red sweets and 7 yellow sweets. If a sweet is taken at random from the box. The sweet is going to be either a red or a yellow sweet. Taking the event of choosing a red sweet to be R and the event of choosing a yellow sweet to be Y. Event R and Y cannot occur at the same time, then we say these events are mutually exclusive events.

What is the probability that the sweet chosen is

a) yellow b) red c) red or yellow

Solutions

a) Probability of choosing a yellow sweet is $P(Y)=\frac{7}{12}$

b) Probability of red sweet is $P(R) =\frac{5}{12}$

c) Probability of Red or yellow $=P(R \text{ or } Y)$

$$=P(R)+P(Y)$$

$$=\frac{5}{12}+\frac{7}{12}$$

$$=1$$

Now, Let us look at another example.

Worked Example [4]

Questions

At a school development meeting there are 10 men and 15 women. One person is chosen to be the chairperson. What is the probability that the person chosen is

(a) a man

(b) a woman

(c) either a man or a woman.

Solutions

a) P (man) $=\frac{10}{25}=\frac{2}{5}$

b) P (woman)$=\frac{15}{25}=\frac{3}{5}$

c) P (man or woman) $=\frac{10}{25}+\frac{15}{25}$

$=\frac{25}{25}=1$

Now that you have gone through this example of calculating the probability of mutually exclusive events, attempt the following activity.

Activity (*21.2*) Mutually exclusive events

Questions

1) A group of people is nominated to be elected as councillor for ward 9. 10 people are from village A, 15 people are from village B and 11 people from village C. If 1 person becomes the overall winner to be the councillor. What is the probability that the councillor is from
   a) Village A
   b) Village B
   c) Village C
   d) either from village A or village B
   e) either from village A nor village C

2) A card is chosen from a pack of playing cards. What is the probability that the card chosen is
   a) either a heart or a spade
   b) either a queen or a jack

3) If ξ ={ letters of the alphabet}

A = {vowels}

B = {b; c; d; p}

If a letter is chosen at random from the letters of the alphabet. What is the probability that the letter is from
   a) A
   b) B
   c) either A or B

4) At a Christian conference there were 40 foreign delegates that is 12 from Malawi, 18 from Tanzania and 10 from Zambia. A foreigner is chosen at random to give a testimony. What is the probability that the person chosen is from

a) Malawi
b) Tanzania
c) either Tanzania or Zambia

Answers:

1.(a) $\frac{5}{18}$ (b) $\frac{5}{12}$ (c) $\frac{11}{36}$ (d) $\frac{25}{36}$ (e) $\frac{5}{12}$

2.(a) $\frac{1}{2}$ (b) $\frac{2}{13}$

3.(a) $\frac{5}{26}$ (b) $\frac{2}{13}$ (c) $\frac{9}{26}$

4.(a) $\frac{3}{10}$ (b) $\frac{9}{20}$ (c) $\frac{7}{10}$

Tip: For mutually exclusive events, the probability of getting A, B, or C is found by summing up the probabilities

P (A or B or C) = P (A) + P (B) +P (C)

21.6 INDEPENDENT EVENTS

When a coin and a die are thrown at the same time, the possible outcomes of a coin do not affect the possible outcomes of the die. Events of this nature are called independent events. A Head and a 5 from a die can occur at the same time that is why they are called independent events.

The outcomes of a die and a coin can be put in a table as shown below

Table 1: Independent events

	Dice					
Coin	1	2	3	4	5	6
Head (H)	H1	H2	H3	H4	H5	H6
Tail (T)	T1	T2	T3	T4	T5	T6

For independent events A and B, the probability of A and B is found by using the **product law** that is P(A and B) = P(A) × P(B)

Tip: If events are performed without mentioning the term replacement, then it is assumed that the events are performed without replacement.

Worked Example [5]

Questions

Using the information in Table 1, find the probability of getting a Head and a 5

Solutions

P (Head) $=\frac{1}{2}$ P (5) $= \frac{1}{6}$

P(Head and 5) = P(Head) × P(5)

$=\frac{1}{2} \times \frac{1}{6}$

$=\frac{1}{12}$

Worked Example [6]

Questions

Mr Moyo had 10 sweets in his pocket, 6 blue and 4 green. He took a sweet from his pocket at random and gave the sweet to his daughter Thelma. He took another sweet and gave it to Tedious his son.

What is the probability that

a) Thelma was given a
   (i) blue sweet
   (ii) Green sweet

b) Both Thelma and Tedious were given
   (i) blue sweets
   (ii) Sweets of different colours

Solutions

(a) i) P(blue) $=\frac{6}{10}=\frac{3}{5}$

ii) P(Green) $=\frac{4}{10}=\frac{2}{5}$

(b) i) Probability of both sweets being blue.

= P(blue and blue)

$=\frac{6}{10}\times\frac{5}{9}$ Note that Tedious was given a sweet when 9 sweets were remaining while 5 sweets are blue.

$=\frac{1}{3}$

ii) Sweets of different colours = blue and green or green and blue.

P(blue and green) +P(green and blue)

$\frac{6}{12}\times\frac{4}{9}+\frac{4}{10}\times\frac{6}{9}$

$\frac{4}{15}+\frac{4}{15}=\frac{8}{15}$

After going through the example of calculating probability of independent events, attempt the following activity.

Activity (*21.3*) Independent events

Questions

1) The following are shoe sizes in a family of 6 {4, 5 ,7 ,7 ,7}.Two people are chosen from the family one after the other what is the probability the two people chosen put on size

a) 7 b) 5

c) The first size 5 and the second size 7

2) The table below represents ages of students in a class

Age in years	16	17	18	19
Number of students	8	5	4	3

If a one person is chosen to be the class monitor and the other vice, what is the probability that two are

a) 16 years
b) 18 years
c) The first is above 17 years and the other below 17 years

Answers

1 (a) $\frac{3}{10}$ (b) 0 (c) $\frac{3}{20}$

2 (a) $\frac{14}{95}$ (b) $\frac{3}{95}$ (c) $\frac{14}{95}$

21.7 COMBINED EVENTS

When two or more events are combined special note should be on the possible outcomes. Pick 10 small stones that are identical in size, 4 of the stones mark them red and the other 6 mark them black. Put the 10 stones in a hat. Take the stones from the hat one after the other.

There are two ways of taking those stones

(i) You take the first stone check the colour and return it, this is called with replacement.

(ii) You take the first stone you check its colour and you do not return it in the hat, this is called without replacement.

Worked Example [7]

Questions

If two stones are chosen one after the other from a set of 4 red and 6 black (choosing with replacement), what is the probability that 2 stones are

a) red
b) black
c) red and black

Solutions

Probability that

a) both red $=\frac{4}{10}\times\frac{4}{10}=\frac{16}{100}$

$=\frac{4}{25}$

b) both black $=\frac{6}{10}\times\frac{6}{10}$

$=\frac{36}{100}$

$=\frac{9}{25}$

c) Red and black or black and red depending with the colour

Starts = P(red and black or black and red)

$=\frac{4}{10}\times\frac{6}{10}+\frac{6}{10}\times\frac{4}{10}$

$=\frac{24}{100}+\frac{24}{100}=\frac{48}{100}=\frac{12}{25}$

Worked Example [8]

Questions

When the two stones are chosen without replacement, what is the probability that

a) Both are read

b) Both are black

c) One is red and the other is black

d) both are red and black or black and red

Solutions

a) Probability that both are red $=\frac{4}{10}\times\frac{3}{9}=\frac{2}{15}$

b) Probability that both are black $=\frac{6}{10}\times\frac{5}{9}=\frac{1}{3}$

c) Probability that one is red and the other black $=\frac{4}{10}\times\frac{6}{9}+\frac{6}{10}\times\frac{4}{9}=\frac{8}{15}$

d) Probability that both are red and black or black and red $=\frac{4}{15}+\frac{4}{15}=\frac{16}{225}$

Now that we have gone through examples of probability of combined events involving choosing with and without replacement, you can now attempt the following activity.

Activity (*21.4*) Combined events

Questions

1) A new farmer has 10 cattle and 5 goats. He chose what to slaughter at random at the birthday party of his daughter.
   a) What is the probability of slaughtering
      (i) cattle
      (ii) goat

   b) If on the day of the party, the farmer chose to slaughter two animals, what is the probability of slaughtering
      (i) 2 cattle
      (ii) first cattle and then goat

2) At a meeting there are 12 men and 8 women. Two people are chosen at random to be the chairperson and vice chairperson.
   What is the probability that they are?
   a) both men
   b) one is a man and the other a woman

Answers:

1 (a) (i) $\frac{2}{3}$ (ii) $\frac{1}{3}$
  (b) (i) $\frac{3}{7}$ (ii) $\frac{5}{21}$

2 (a) $\frac{33}{95}$ (b) $\frac{48}{95}$

21.8 OUTCOME TABLES.

Outcome tables help us to come out with the possible outcomes of combined events. For example when a coin is tossed 2 times, the possible outcomes can be shown as an outcome table shown below:

	H	T
H	HH	HT
T	TH	TT

There are 4 possible outcomes.

From the table above probability of getting a head and a tail = $\frac{2}{4} = \frac{1}{2}$

The same applies when calculating probability of two heads = P (H H) =$\frac{1}{4}$

Worked Example [9]

Questions

Two dice are thrown at the same time and the results are added.

a) Draw an outcome table for the two dice.

b) How many possible outcomes are there?

c) Find the probability of getting a

(i) prime number

(ii) perfect square

(iii) perfect cube

(iv) multiples of 5.

Solutions

1st Die

2nd Die

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a) There are 36 possible outcomes.

b) Probability of prime number

(i) P(prime number) = $\frac{15}{36}$ (number of prime numbers)

$= \frac{5}{12}$

(ii) P(perfect square) = $\frac{7}{36}$ (number of perfect squares)

(iii) P(perfect cubes) $= \frac{5}{36}$

(iv) P(multiples of 5) $= \frac{7}{36}$

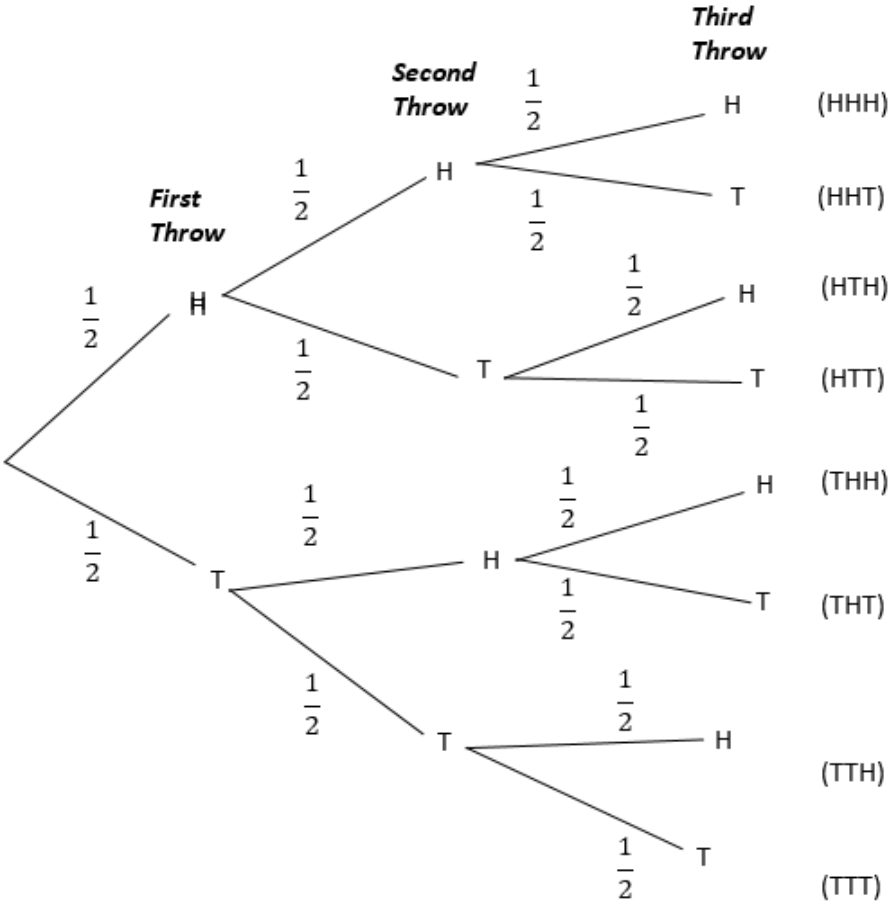
21.9 TREE DIAGRAM

Just like outcome tables, tree diagrams are used to establish the possible outcomes of two or more consecutive events. Each branch represents an event.

Worked Example [10]

Questions

A coin may be tossed three times, the outcomes can be shown in a diagram below.



Using the diagram, find the probability of getting

a) 3 heads

b) 2 heads

c) At least two tails.

Solutions

a) Probability of 3 heads, you take the branch that has 3 heads (H H H)

Therefore P (H H H) $=\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

b) Probability of 2 heads = H H T or H T H or T H H

P (H HH T) $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

P (H T H) $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

P (T H H) $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

The probability of 2 heads $= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$= \frac{3}{8}$

c) Probability of at least 2 tails, check for combination with 2 tails or more H T T or T H T or T T H or T TT

P (H T T or T H T or T T H or T T T)

$= (\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2})$

$\frac{1}{8} \quad + \quad \frac{1}{8} \quad + \quad \frac{1}{8} \quad + \quad \frac{1}{8}$

$= \frac{4}{8} = \frac{1}{2}$

Worked Example [11]

Questions

A bag contains 12 balls, 5 red and 7 black. Two balls are chosen from the bag one after the other without replacement.

a) Draw a tree diagram to show how the balls are chosen.

b) What is the probability that the two balls are

(i) of the same colour

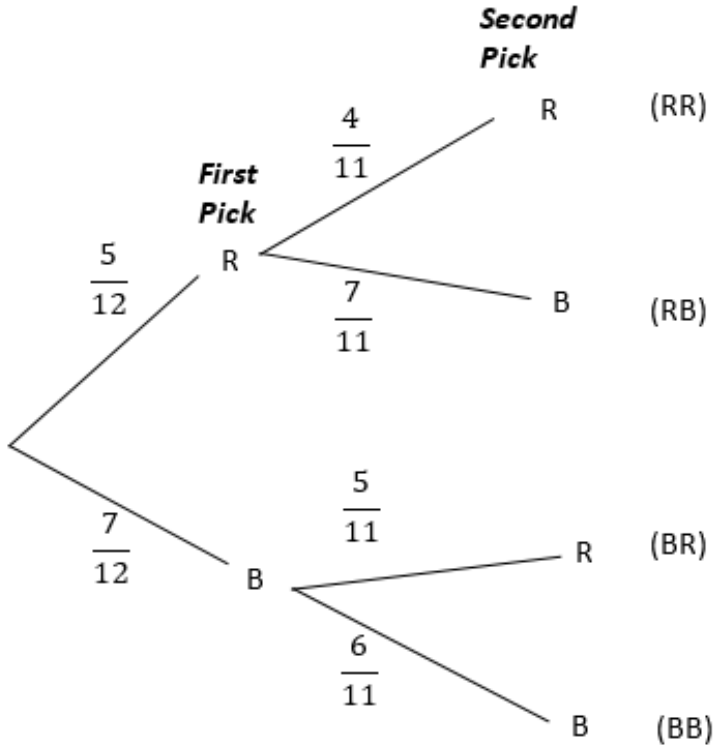
(ii) of different colours

(iii) the first is black and red.

Solutions

Let red be R and black be B

a)



b) P(same colour) = P(R R or BB) = P(R R) + P(BB)

$= \frac{5}{12} \times \frac{4}{11} + \frac{7}{12} \times \frac{6}{11}$

$= \frac{5}{33} + \frac{7}{22}$

$= \frac{6}{11}$

c) P(different colours)

=P (RB or BR)

$= \frac{5}{12} \times \frac{7}{11} + \frac{7}{12} \times \frac{5}{11}$

$= \frac{35}{132} + \frac{35}{132}$

$= \frac{70}{132}$

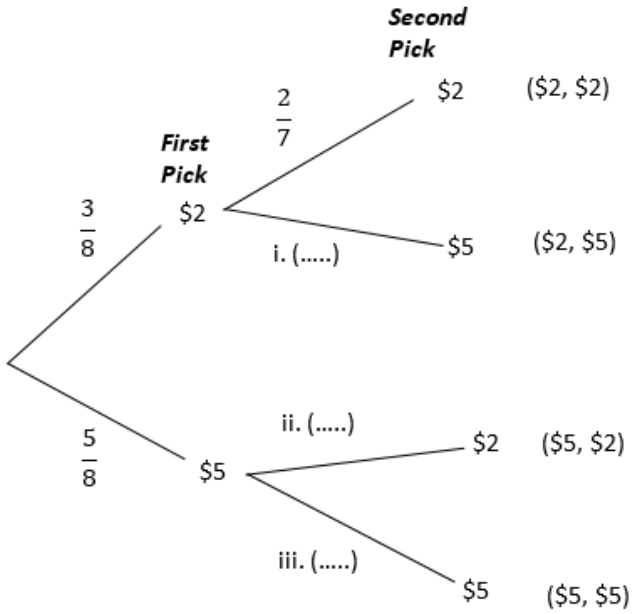
$= \frac{35}{66}$

After going through the above examples, you should be now in a position to attempt the following activity.

Activity (*21.5*) Tree diagrams

Questions

1. Mr Moyo has five $5 notes and three $2 notes in his wallet. He takes 2 notes at random from the wallet one after the other and gave to his daughter as a present for doing well at school. What is the probability that, he gave his daughter
   a) $ 4
   b) $ 7
   c) Complete the tree diagram below.



2. Peter has 5 balls in his pocket written the following numbers {2, 3, 5, 13}. He takes two balls from the pocket one after the other with replacement .What is the probability that
   a) the product is 49
   b) The product is 10

3. Chipo is invited for an interview as a bank teller where she has to first take an oral test and then a written test. To proceed to written test she has to pass oral test.

Given that the probability of passing oral test is 0,7 and the written is 0,4. What is the probability that she

a) passed all the tests
b) passed the oral and failed the written
c) did not proceed to written test

Answers

1 (a) $\frac{3}{28}$ (b) $\frac{15}{28}$ (c) i. $\frac{5}{7}$ ii. $\frac{3}{7}$ iii. $\frac{4}{7}$

2. (a) 0 (b) $\frac{2}{25}$

3. (a) 0.28 (b) 0.42 (c) 0.3

21.10 Summary

Now that you are now able to define probability and other terms used in probability, you can now solve problems involving probability.

You are now also able to use outcome tables and tree diagrams to come up with possible outcomes .You should know when to add or multiply probability.

Remember probability is a measure between 0 and 1 that is $0 \leq p \leq 1$.Try the assessment exercise below then move on to the next unit.

21.11 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

21.12 Assessment Test

1. A bag contains 3 blue balls, 4 green balls and 5 white balls. A ball is taken from the bag at random. What is the probability that it is
   a) blue (1)
   b) green (1)
   c) white (1)
   d) black (1)
   e) either blue or green (2)
   f) neither blue nor green (2)

2. The table below represents ages of students in a school bus.

Age in years	13	14	15	16	17
Number of students	4	5	6	8	2

   a) How many students were in the bus? (1)
   b) If one student is chosen from the bus ,what is the probability that the student is
      (i) 16 years old (1)
      (ii) 13 or 14 years old (1)
      (iii) more than 14 years old (1)

3. A box of pen contains 4 blue pens and 6 black pens, two pens are chosen at random from the box. What is the probability that pens are
   a) both blue (2)
   b) of different colours (2)
   c) at least one is blue (2)

4. The probability that Chipo will pass an examination is $\frac{5}{8}$ and for Susan is $\frac{x}{4}$
   a) If the probability that both will pass the examination is $\frac{5}{16}$ find x (2)
   b) Use the value of x to find the probability that one of the two will pass the examination. (3)

5. You are herding 40 cattle and 10 goats .Two of the animals went astray .What is the probability that they are
   a) (a) both cattle (2)
   b) (b) cattle and goat (2)

Answers

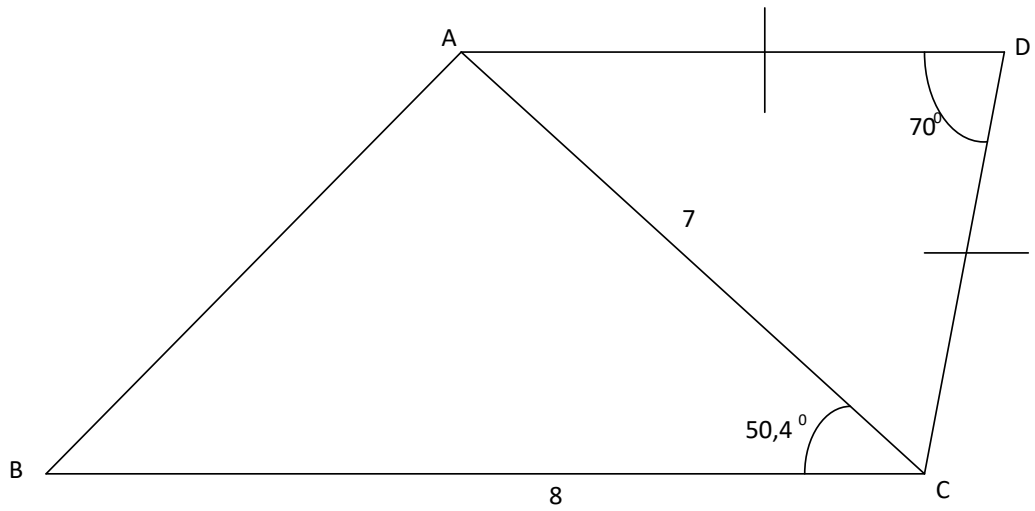
1. (a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{5}{12}$ (d) 0 (e) $\frac{7}{12}$ (f) $\frac{5}{12}$
2. (a) 25 (b) (i) $\frac{8}{25}$ (ii) $\frac{9}{25}$ (iii) $\frac{16}{25}$
3. (a) $\frac{2}{15}$ (b) $\frac{8}{15}$ (c) $\frac{2}{3}$
4. (a) 2 (b) $\frac{1}{2}$
5. (a) $\frac{156}{245}$ (b) $\frac{16}{49}$

Sample Examination Questions.

1) From the top of an Econet booster, the angle of depression of a mouse on level ground is 36.4°.Given that the booster is 15m in height, Calculate the distance of the mouse from the foot of the booster (3)
   You may use the information below
   Sin 36,4° = 0,59 Cos 36,4° = 0,80 Tan 36,4° = 0,74
   Sin 53,6° = 0,80 Cos 53,6° = 0,59 Tan 53,6° =1,36

2) (a) The bearing of village B from village A is 240°. What is the three figure bearing of village from village B (2)

(b)



ABCD is a quadrilateral, BC = 8cm, AC = 7cm AD = DC, $A\hat{C}B$ = 50.4° and ADC = 70°. Using as much of the information given below as necessary

Calculate

(i) area of triangle ABC (3)

(ii) Angle DAC (2)

(iii) Length of AD (3)

(iv) Length of AB (3)

(v) Shortest distance from A to BC (2)

Sin 50,4° = 0,77	cos 50,4 = 0,64°	tan 50,4° = 1,21
Sin 70° = 0,94	cos 70° = 0,34°	tan 70° = 2,75
Sin 55° = 0,82	cos 82 = 0,57	tan 55° = 1,43

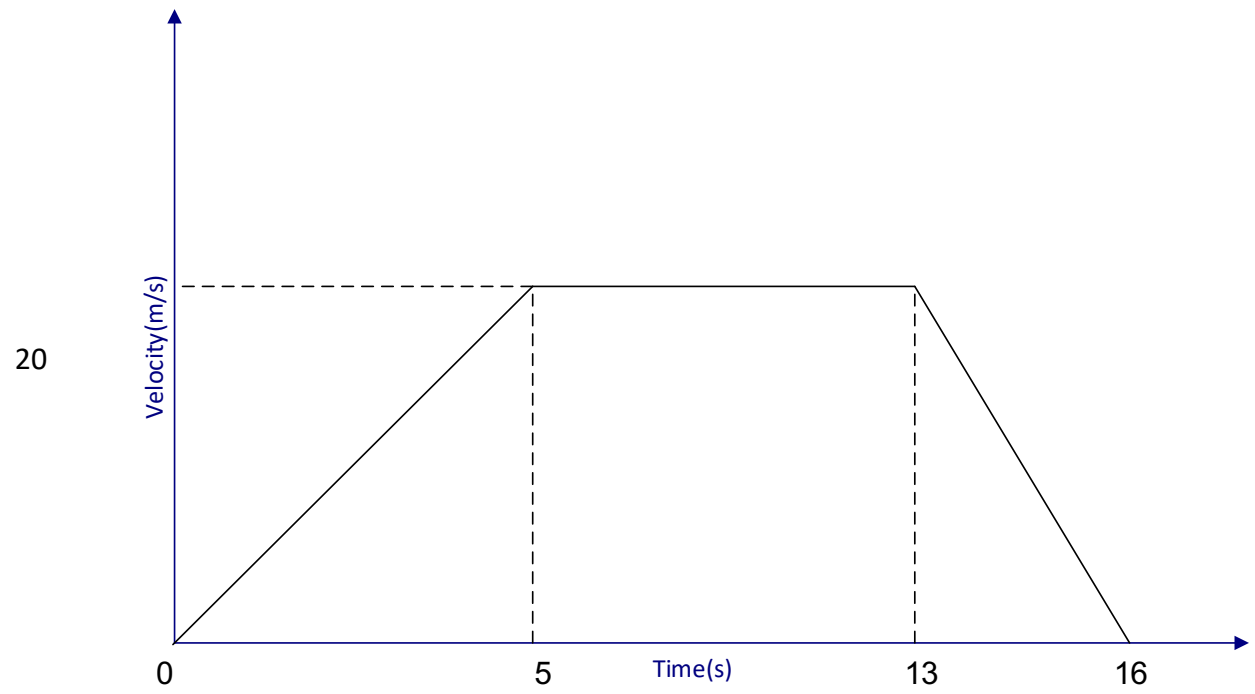
3) A box contains 5 red sweets 3 yellow sweets which are identical except for colour. John takes a sweet from the box at random and eats it.
David then takes a sweet from the box at random Giving your answers as fractions in their lowest terms calculate the probability that

a) John takes a red sweet (1)

b) both John and David take red sweet (1)

c) both John and David take sweets of the same colour (2) ZIMSEC 1997.

4.



The diagram is velocity time graph of an object which accelerates uniformly for 5 seconds to a velocity of 20m/s. It moves with constant velocity for 8 seconds. Calculate

a) the acceleration in the first 5 seconds (1)
b) the velocity of the object at t = 3 (2)
c) the total distance covered in 16 seconds (2)
d) the average speed of the object. (2)

Answers :

1. 20.3m
2. (a) 060° (b)(i) 21.56cm^2 (ii) 55° (iii) 6.10cm (iv) 6.43cm (v) 5.39cm
3. (a) $\frac{5}{8}$ (b) $\frac{5}{14}$ (c) $\frac{13}{28}$
4. (a) 4m/s^2 (b) 12m/s (c) 240m (d) 15m/s

UNIT 22 - VECTORS

CONTENTS

22.1 INTRODUCTION

In this unit we introduce vectors from the basics despite the fact that you might have done something on vectors at Level 1. In order to understand the concepts involved in Vectors, you should have understood previous concepts on the Cartesian plane and also on Plane shapes. These two concepts are essential in the understanding of translation vectors and properties of shapes in vectors.

OBJECTIVES

After going through this unit, you should be able to

- describe types of vectors
- solve problems involving vector operations

KEY WORDS

Vector: – a quantity with direction and size (magnitude or modulus) e.g. velocity, acceleration, displacement, force as well as translation

Scalar:– a quantity that has size only or a numerical value that multiplies, reduces or stretches a vector or a matrix

Translation:– it is a vector quantity which involves movement in a straight line and without turning

TIME:- You are expected not to spend more than 8 hours in this unit.

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems of Vectors as possible for you to grasp all the concepts in this topic.

22.2 TYPES OF VECTORS

Let us start by looking at vector notation before we get into the types of vectors.

22.2.1 Vector notation

Vectors can be represented as follows

- In letters $\overrightarrow{AB}$ or $\overline{AB}$ or $\mathbf{AB}$ or $\underline{AB}$ or $\vec{a}$ or $\bar{a}$ or $\mathbf{a}$ or $\underline{a}$
- a column matrix $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$

NOTE: Direction is very important in vector notation

22.2.2 Translation vectors

Normally, a translation vector is denoted by a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$, where x and y represent movements taken along the x-axis and the y-axis respectively. It should be noted that movements to the right and upwards along the respective axes are positive, while movements to the left and downwards parallel to the respective axes are negative.

Worked Example [1]

Questions

The diagram below shows translation vectors. Write these vectors in the form $\begin{pmatrix} x \\ y \end{pmatrix}$

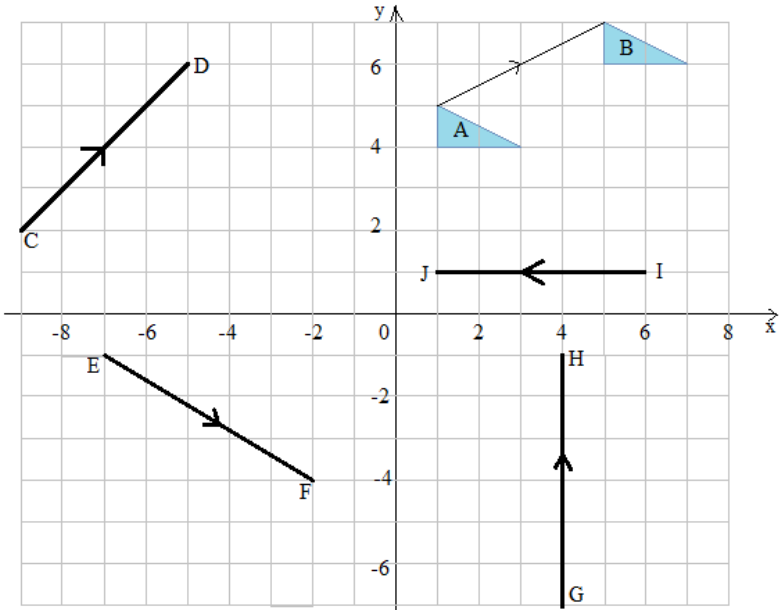


Fig 22.1

Solutions

Method 1(by counting units)

Follow the direction of the arrow and count units along the x-axis first and continue counting the units along the y-axis

$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$, $\overrightarrow{EF} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, $\overrightarrow{GH} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$, $\overrightarrow{IJ} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$

Method 2(by subtracting coordinates)

The coordinates are A(1;5), B(5;7), C(-9;2), D(-5;6),E(-7;-1),F(-2;-4), G(4;-7), H(4;-1) , I(6;1) and J(1;1)

$\overrightarrow{AB} = B - A = \begin{pmatrix} 5 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ as a column vector

$\overrightarrow{CD} = D - C = \begin{pmatrix} -5 \\ 6 \end{pmatrix} - \begin{pmatrix} -9 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$\overrightarrow{EF} = F - E = \begin{pmatrix} -2 \\ -4 \end{pmatrix} - \begin{pmatrix} -7 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

$\overrightarrow{GH} = H - G = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ -7 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

$$\overrightarrow{IJ} = J - I = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$$

Activity (*22.1*) Translation vectors

Questions

1) In the diagram below are line segments representing translation vectors. Write these vectors in the form $\begin{pmatrix} x \\ y \end{pmatrix}$
2) If an object is translated from point A(-3;7) to point B(4;6). Find
   (a) $\overrightarrow{AB}$
   (b) $\overrightarrow{BA}$
3) Given that $\overrightarrow{XY} = \begin{pmatrix} -8 \\ 11 \end{pmatrix}$ and the point X is (5;-1), find the coordinates of point Y
4) If $\overrightarrow{PQ} = \begin{pmatrix} x \\ -3 \end{pmatrix}$, P(2;y) and Q(-12;4), find the values of x and y

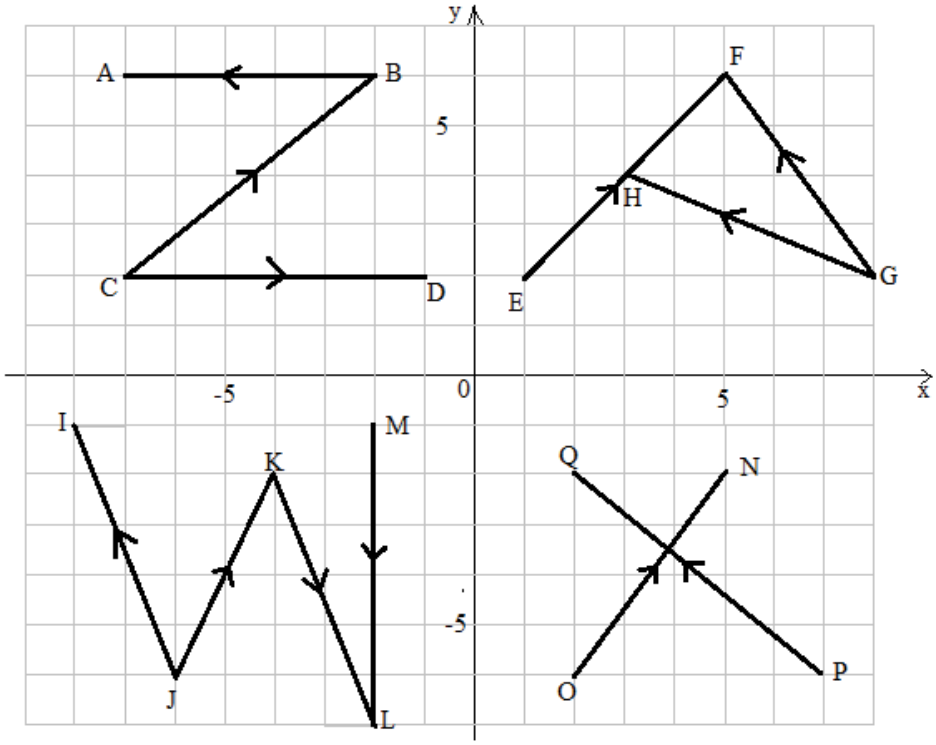


Fig 22.2

Answers

1. **BA** = $\binom{-5}{0}$, **CB** = $\binom{5}{4}$, **CD** = $\binom{6}{0}$, **EF** = $\binom{4}{4}$, **EH** = $\binom{2}{2}$, **GF** = $\binom{-3}{4}$, **GH** = $\binom{-5}{2}$, **JI** = $\binom{-2}{5}$, **JK** = $\binom{2}{4}$, **KL** = $\binom{2}{-5}$, **ML** = $\binom{0}{-6}$, **ON** = $\binom{3}{4}$, **PQ** = $\binom{-5}{4}$
2. (a) $\overrightarrow{AB} = \binom{7}{-1}$ (b) $\overrightarrow{BA} = \binom{-7}{1}$
3. Y(–3;10)
4. x = –14 and y = 7

22.2.3 Equal vectors

These are vectors showing the same direction and the same magnitude.

Figure 22.4 below shows equal vectors

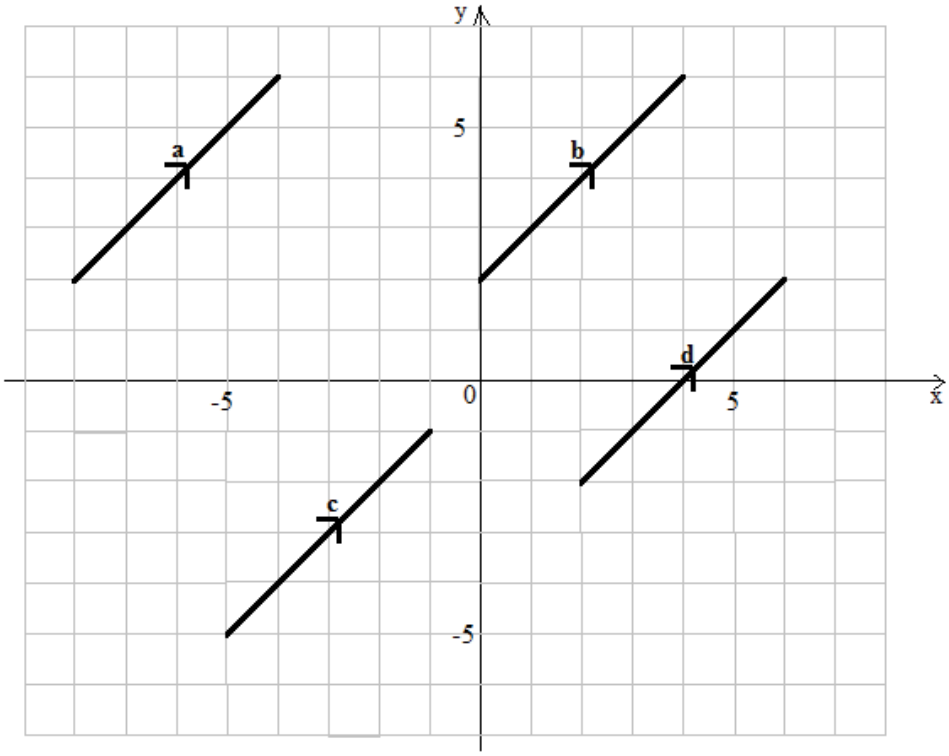


Fig 22.4

It should be noted that besides the different positions of the vectors, the four vectors are equal, that is, **a = b = c = d =** $\binom{4}{4}$

22.2.4 Negative Vectors

A negative vector is a vector whose direction has been reversed which means that the direction has been changed. However, changing the direction of a vector does not affect the size or magnitude of the vector.

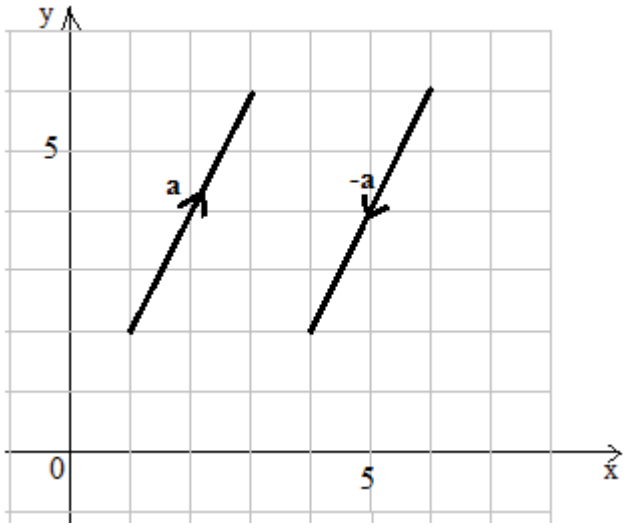


Fig 22.6

In figure 22.6 above **a** = $\binom{2}{4}$ and **–a = -(a) =** $-\binom{2}{4}$ **=** $\binom{-2}{-4}$**.** In negative vectors we are literally applying a negative sign to reverse a vector.

22.2.5 Parallel vectors

Parallel vector show equality in terms of direction despite the size of a vector.

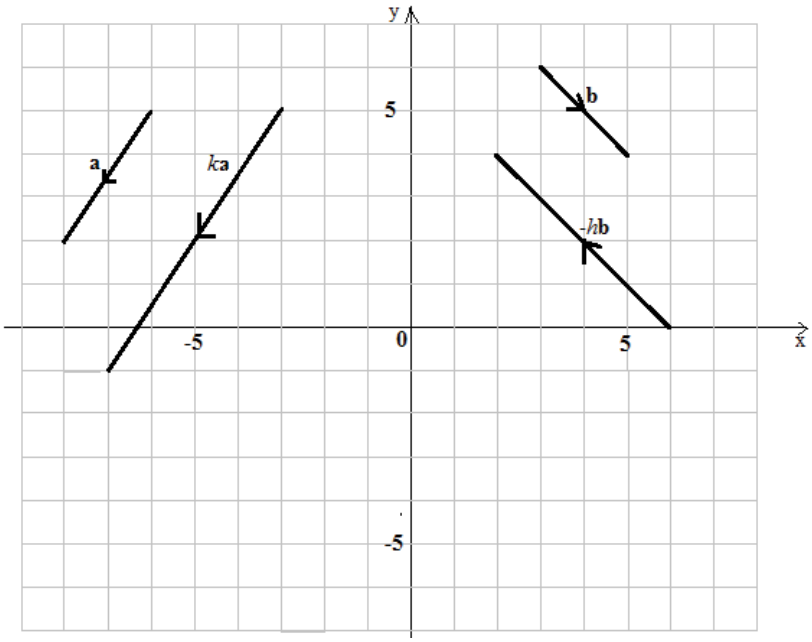


Fig 22.7

In figure 22.7 above **a** = $\begin{pmatrix}-2\\-3\end{pmatrix}$ is parallel to k**a** = $\begin{pmatrix}-4\\-6\end{pmatrix}$= 2$\begin{pmatrix}-2\\-3\end{pmatrix}$, where k = 2(scalar). The scalar represents the scale factor of enlargement of vector **a.**

Worked Example [2]

Questions

Given that **q** = $\begin{pmatrix}2\\5\end{pmatrix}$ and **p** = $\begin{pmatrix}x\\15\end{pmatrix}$ are parallel vectors. Find the value of x

Solutions

q = k**q**

k**q** = **p,** that is,

$k\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}x\\15\end{pmatrix}$

$\begin{pmatrix}2k\\5k\end{pmatrix} = \begin{pmatrix}x\\15\end{pmatrix}$

$5k = 15$

$k = 3$

$x = 2k$

therefore, $x = 2(3) = 6$

NOTE: Cross multiply the components of the vector, thus

$\begin{pmatrix}2\\5\end{pmatrix}\begin{pmatrix}x\\15\end{pmatrix}$

$2 \times 15 = 5 \times x$

$5x = 30$

$x = 6$

22.2.6 Position vectors

A position vector is a vector that is directly linked to the origin O(0;0), thus if point P(x; y) gives a position vector $\overrightarrow{OP} = \begin{pmatrix}x\\y\end{pmatrix}$

Worked Example [3]

Questions

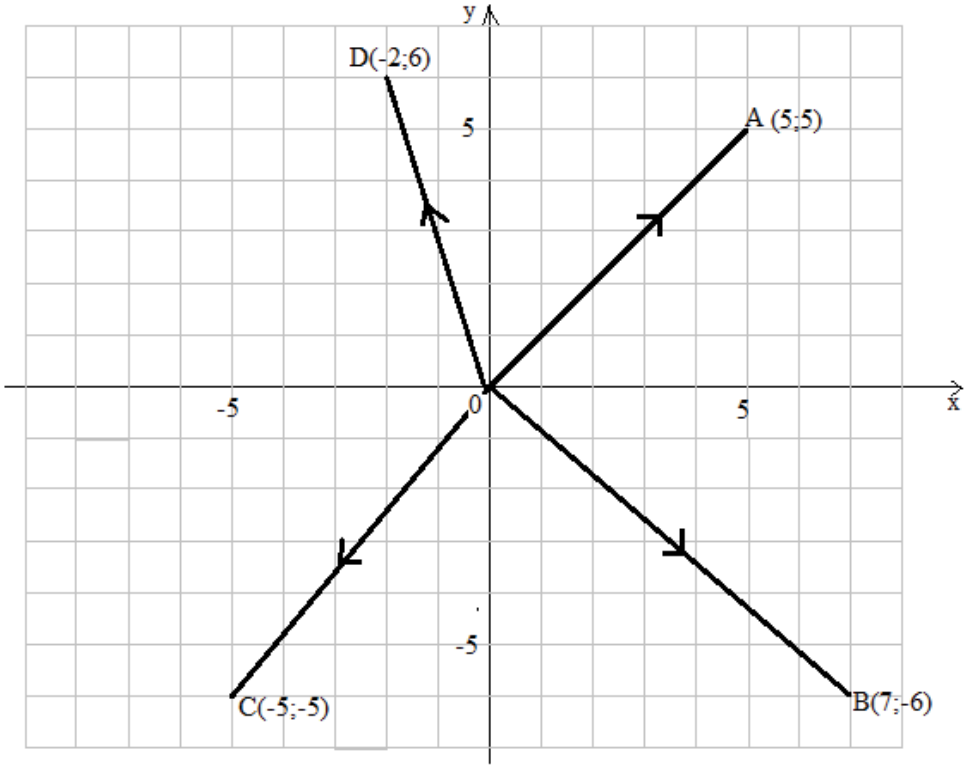


Fig 22.8

Solutions

The points A,B,C and D are directly linked to the origin O and the position vectors are

$\overrightarrow{OA} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 7 \\ -6 \end{pmatrix}, \overrightarrow{OC} = \begin{pmatrix} -5 \\ -6 \end{pmatrix}$ and $\overrightarrow{OD} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

Activity (22.2) Equal, negative, parallel, position vectors

Questions

1. Given that $\overrightarrow{OY} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, $\overrightarrow{OZ} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and M is the midpoint of YZ. Find

(a) $\overrightarrow{YZ}$ (b) $\overrightarrow{OM}$ (c) $\overrightarrow{MY}$ (d) $\overrightarrow{ZY}$

2. Given that $\overrightarrow{OA} = 2\,\overrightarrow{AB}$, $\overrightarrow{OA} = \begin{pmatrix} -8 \\ 4 \end{pmatrix}$ and X is the midpoint of AB. Find,

(a) $\overrightarrow{AB}$ (b) $\overrightarrow{OX}$ (c) $\overrightarrow{AX}$ (d) $\overrightarrow{BA}$

3. If $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\overrightarrow{SR} = \begin{pmatrix} 7 \\ x \end{pmatrix}$ are parallel.

(a) Find x

(b) Express $\overrightarrow{SR}$ in the form $k\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, where k is a constant.

4. Given that a parallelogram has coordinates O(0;0), A(4;0), B(6;7) and C(x;y). Find

(a) the coordinates of point C

(b) $\overrightarrow{AB}$

(c) $\overrightarrow{OA}$

(d) $\overrightarrow{OB}$

(e) $\overrightarrow{OC}$

Solutions

1. (a) $\begin{pmatrix} -9 \\ 5 \end{pmatrix}$, (b) $\begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$ (c) $\begin{pmatrix} 4.5 \\ -2.5 \end{pmatrix}$, (d) $\begin{pmatrix} 9 \\ -5 \end{pmatrix}$

2. (a) $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} -10 \\ 5 \end{pmatrix}$ (c) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$

3. (a) $x = 9\frac{1}{3}$ (b) $\frac{7}{3}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

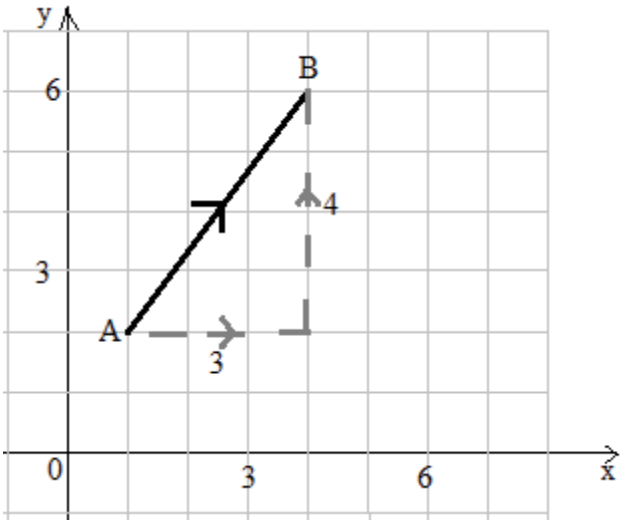
4. (a) C(2;7) (b) $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ (c) $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ (d) $\begin{pmatrix} 6 \\ 7 \end{pmatrix}$ (e) $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$

22.3 MAGNITUDE OF VECTORS

This refers to the size or modulus of a vector. If $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $|\overrightarrow{AB}| = \sqrt{x^2 + y^2}$ (by Pythagoras theorem)

Worked Example [4]

Questions



The diagram above shows how the Pythagoras theorem is linked to the concept of magnitude of vectors.

In the diagram, $\overrightarrow{AB} = \binom{3}{4}$, find $\overrightarrow{|AB|}$

Solutions

$\overrightarrow{|AB|} = \sqrt{3^2 + 4^2}$

= 5 Units (by Pythagoras theorem since 3, 4 and 5 make a Pythagorean triple)

22.4 COMBINED VECTORS

22.4.1 Sum of vectors

In this section we explain addition and subtraction of vectors.

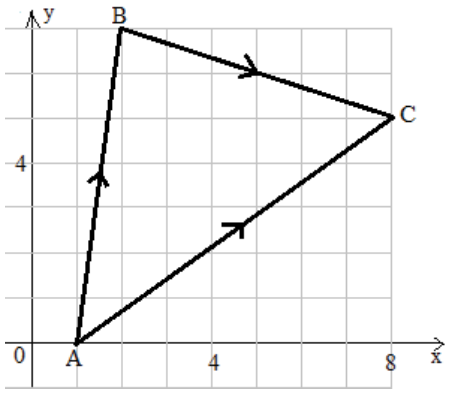


Figure 22.11

Figure 22.3 shows three connected vectors, that is, $\overrightarrow{AB} = \binom{1}{7}$, $\overrightarrow{BC} = \binom{6}{-2}$ and $\overrightarrow{AC} = \binom{7}{5}$. Since these vectors are connected, the resultant vector $\overrightarrow{AC}$ is given by $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (**Law of vector addition**), thus,

$$\overrightarrow{AC} = \binom{1}{7} + \binom{6}{-2} = \binom{7}{5}$$

Worked Example [5]

Questions

Given that $\overrightarrow{PQ} = \binom{-3}{-5}$ and $\overrightarrow{QR} = \binom{7}{9}$, find $\overrightarrow{PR}$

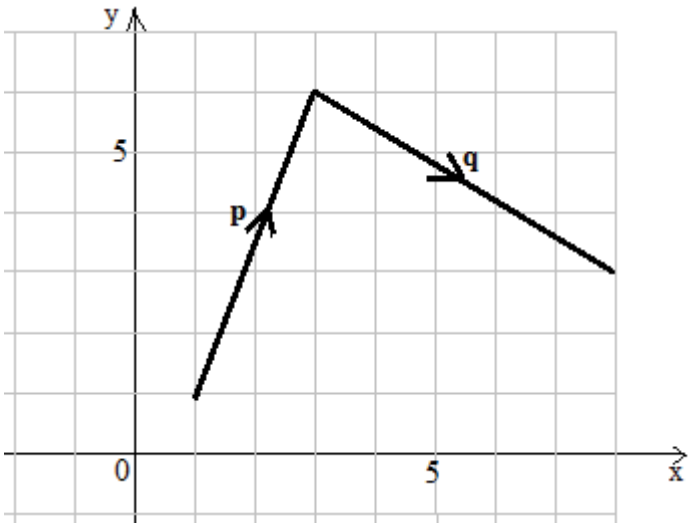
Solutions

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$$

$$\overrightarrow{PR} = \binom{-3}{-5} + \binom{7}{9} = \binom{4}{4}$$

Activity (*22.3*) Magnitude of vectors and combined vectors

Questions



1) Using the diagram, Find
   (a) vectors **p** and **q**
   (b) **p + q**
   (c) **– p – q**

(d) $|\mathbf{p}|$
(e) $|\mathbf{q}|$
(f) $|\mathbf{p}|+|\mathbf{p}|$
(g) $|\mathbf{q}|+|\mathbf{q}|$
(h) $|\mathbf{p}|+|\mathbf{q}|$
(i) $|\mathbf{p}+\mathbf{q}|$
(j) $|\mathbf{p}-\mathbf{q}|$
(k) $|\mathbf{p}|-|\mathbf{q}|$

2) If a = $\begin{pmatrix}-2\\5\end{pmatrix}$, b = $\begin{pmatrix}6\\0\end{pmatrix}$, c = $\begin{pmatrix}5\\-1\end{pmatrix}$ and d = $\begin{pmatrix}0\\8\end{pmatrix}$, find the following

(a) **a + b**
(b) **a + c**
(c) **b** + **d**
(d) **a + b** + **c**
(e) **a** + **b** + **c** + **d**
(f) $|\mathbf{a}+\mathbf{b}|$
(g) $|\mathbf{a}+\mathbf{c}|$
(h) $|\mathbf{b}+\mathbf{d}|$
(i) $|\mathbf{a}+\mathbf{b}+\mathbf{c}|$
(j) $|\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}|$
(k) $|\mathbf{a}+\mathbf{b}|$ **+** $|\mathbf{a}+\mathbf{c}|$
(l) $|\mathbf{a}+\mathbf{b}+\mathbf{c}|$ **-** $|\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}|$

Answers

1. (a) $\mathbf{p}=\begin{pmatrix}2\\5\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}5\\-3\end{pmatrix}$ (b) $\begin{pmatrix}7\\2\end{pmatrix}$, (c) $\begin{pmatrix}-7\\-2\end{pmatrix}$ (d) $\sqrt{29}$ (e) $\sqrt{34}$ (f) $2\sqrt{29}$ (g) $2\sqrt{34}$ (h) $\sqrt{29}+\sqrt{34}$ (i) $\sqrt{53}$ (j) $\sqrt{73}$ (k) $\sqrt{29}-\sqrt{34}$

2. (a) $\begin{pmatrix}4\\5\end{pmatrix}$, (b) $\begin{pmatrix}3\\4\end{pmatrix}$, (c) $\begin{pmatrix}6\\8\end{pmatrix}$, (d) $\begin{pmatrix}9\\4\end{pmatrix}$, (e) $\begin{pmatrix}9\\12\end{pmatrix}$, (f) $\sqrt{31}$ (g) 5 (h) 10 (i) $\sqrt{97}$ (j) 15

22.5 VECTOR PROPERTIES OF PLANE SHAPES

Under this aspect of vectors we use vectors to prove properties of shapes and in reverse the properties can be used to solve some vector problems. Again in this section we move away from the confinements of the Cartesian plane, thus vectors will be used in any situation.

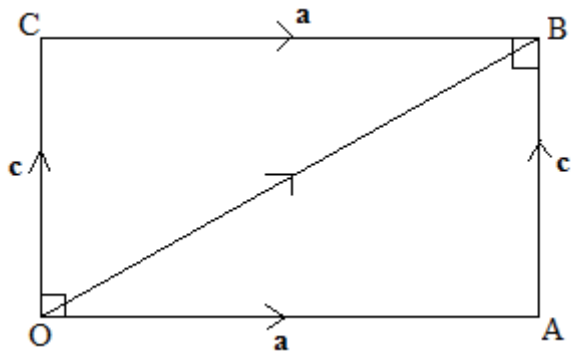


Fig 22.13

In figure 22.23 above OABC is a rectangle. A rectangle is a common shape and the properties have been learnt at various levels even at primary level. It is clear that opposite sides of a rectangle are parallel and equal, and with that we can easily observe that

OA = CB = a and **OC = AB = c.**

Note: If the direction of a vector is reversed then we are required to apply a negative to the existing direction of a vector, that is **AO = - OA = - a** and **CO = - OC = - c**

Finally, some missing vectors can be found by combining given vectors, that is

OB = OA + AB = a + c

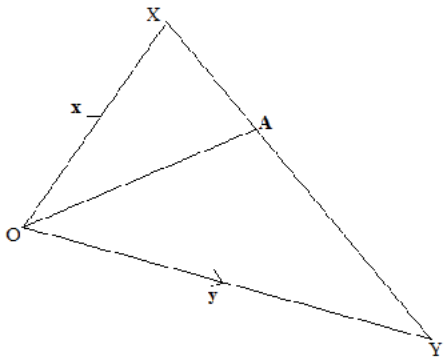
or

OB = OC + CB = c + a

Therefore, **a + c = c + a**

Worked Example [6]

Questions



In the diagram above is a triangle OXY, where **OX = x, OY = y and** XA: AY = 2:5. Express **the** following vectors in terms of **a** and/ or **b.**

(a) **XY** (b) **XA** (c) **AY**

(d) **YA** (e) **OA**

Solutions

(a) **XY = XO + OY**

XY = –x + y = y – x

(b) **XA =** $\frac{2}{7}$ **XY**

XA = $\frac{2}{7}$ **(y – x) =** $\frac{2}{7}$ **y -** $\frac{2}{7}$ **x**

(c) **AY =** $\frac{5}{7}$ **XY**

AY = $\frac{5}{7}$ **(y – x) =** $\frac{5}{7}$ **y -** $\frac{5}{7}$ **x**

(d) **YA =** $\frac{5}{7}$ **(- XY)**

YA = $\frac{5}{7}$ **x -** $\frac{5}{7}$ **y**

(e) **OA = OX + XA**

OA = x + $\frac{2}{7}$ **y -** $\frac{2}{7}$ **x**

OA = $\frac{5}{7}$ **x +** $\frac{2}{7}$ **y**

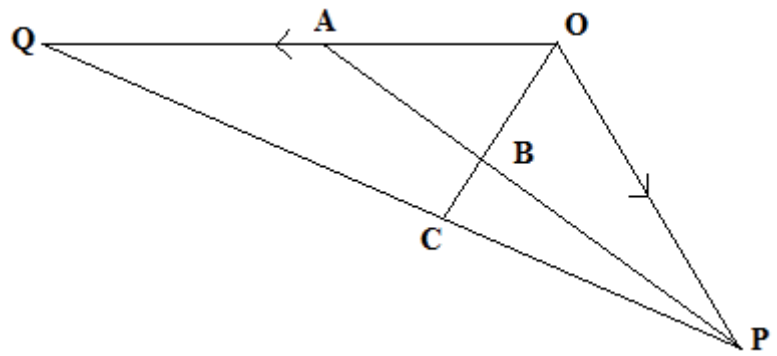
22.6 VECTOR ALGEBRA (FINDING SCALARS)

This aspect falls under properties of shapes and you reminder to revisit or recall the use of simultaneous equations in solving problems, as this is crucial in finding scalars of vectors.

Worked Example [7]

Questions

In the following diagram, A is the midpoint of **OQ,** B is the intersection point of **AP** and **OC** and **PC** = $\frac{2}{5}$**PQ.**



Given that **OP = p** and **OQ = q**

Express the following in terms of **p** and/ or **q**

(a) (i) **PQ** (ii) **OC** (iii) **AP**

(b) Given that **PB** = m**PA**, Find **OB**

(c) Given also that **OB** = n**OC**, using the results in (b) find m and n

(d) Find the numerical value of the ratio $\frac{\mathbf{OB}}{\mathbf{OC}}$

Solutions

(a) (i) **PQ = PO + OQ = PQ = - p + q = q – p**

(ii) **OC = OP + PC = OP +** $\frac{2}{5}$**PQ = p +** $\frac{2}{5}$**(q – p) =** $\frac{2}{5}$**q +** $\frac{3}{5}$**p**

(iii) **AP = AO + OP =** $-\frac{1}{2}$**q + p = p** $-\frac{1}{2}$**q**

(b) **PB** = m**PA**

PB = m($\frac{1}{2}$**q – p) =** $\frac{1}{2}$m**q** – m**p**

Then, **OB = OP + PB = p +** $\frac{1}{2}$m**q** – m**p** = $\frac{1}{2}$m**q + (**1 – m)**p**

(C) OB = n**OC** = n ($\frac{2}{5}$**q +** $\frac{3}{5}$**p**) = $\frac{2}{5}$n**q +** $\frac{3}{5}$n**p**

Then, by comparing equal vectors of **OB,** that is,

OB$_1$ **= OB**$_2$

$\frac{1}{2}$m**q + (**1 – m)**p =** $\frac{2}{5}$n**q +** $\frac{3}{5}$n**p**

Since vectors **q** and **p** are identical we can equate the scalars, that is,

$\frac{1}{2}m = \frac{2}{5}n$ (Scalars of **q**)..... (i)

(1 – m) = $\frac{3}{5}n$ (scalars of **p)** .... (ii)

By solving the two equations simultaneously

Remember (elimination or substitution method)

From equation (i) 5m - 4n = 0

From equation (ii) 5m + 3n = 5

5m + 3n = 5

-[5m - 4n = 0]

7n = 5

$n = \frac{5}{7}$

and

$5m + 3(\frac{5}{7}) = 5$

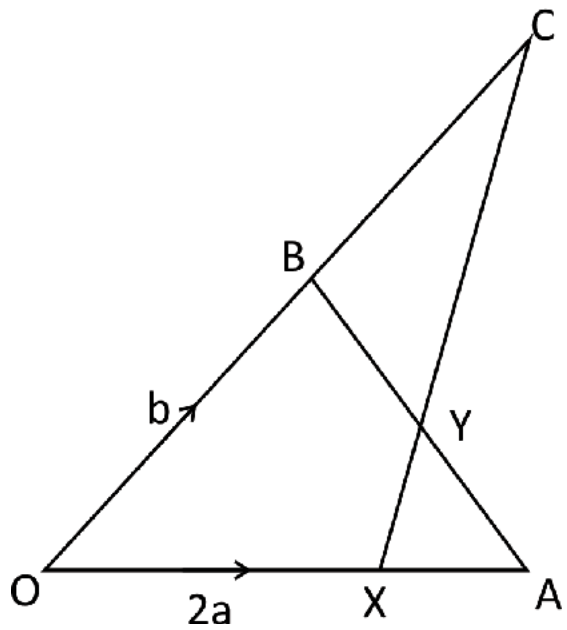
$m = \frac{4}{7}$

(d). $\frac{\mathbf{OB}}{\mathbf{OC}} = \frac{\mathbf{nOC}}{\mathbf{OC}} = n = \frac{5}{7}$

Activity (*22.4*) Properties of shapes and vector algebra

Questions

1. ZIMSEC JUNE 2004 P2 Q7



In the diagram X is a point on OA such that OX:XA = 2:1 and Y is the midpoint of AB. The point C on AB produced is such that OB = BC.
$\overrightarrow{OA} = 3\mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) Express the following vectors in terms of **a** and/or **b**

(a) $\overrightarrow{AB}$,

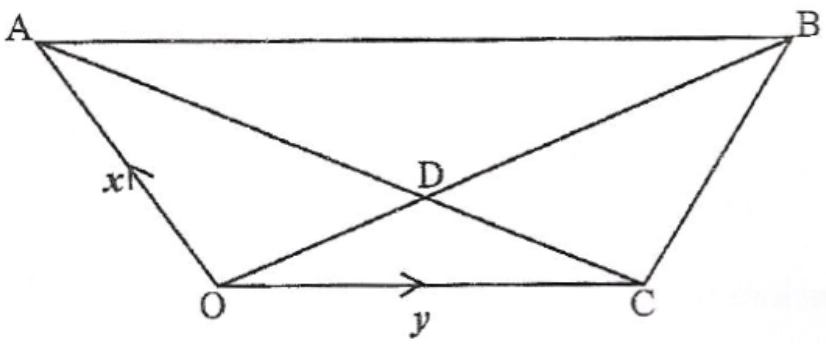
(b) $\overrightarrow{OY}$,

(c) $\overrightarrow{XY}$,

(d) $\overrightarrow{YC}$.

(ii) State the ratio XY:YC. [9]

2. ZIMSEC JUNE 2011 P2 Q7 (b)



The diagram shows a trapezium OABC where OC is parallel to AB, with $\overrightarrow{OA} = x$ and $\overrightarrow{OC} = y$. Diagonals OB and AC intersect at D such that AD : DC = 3 : 2.

Express, in terms of x and/or y,

(i) (a) $\overrightarrow{AC}$,

(b) $\overrightarrow{AD}$.

(ii) Given that $\overrightarrow{AB} = k\overrightarrow{OC}$, express $\overrightarrow{OB}$ in terms of k, x and y.

(iii) Given also that $\overrightarrow{OB} = h\overrightarrow{OD}$, express $\overrightarrow{OB}$ in terms of h, x and y.

(iv) Using results from **(ii)** and **(iii)** above, find the numerical value of h and the numerical value of k.

[7]

Answers

1. (i) (a) $-3\mathbf{a} + \mathbf{b}$ (b). $\frac{3}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$ (c) $\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$ (d) $\frac{3}{2}\mathbf{a} - \frac{3}{2}\mathbf{b}$

(ii) $\frac{1}{3}$

2. (i) (a) $\mathbf{y} - \mathbf{x}$ (b). $\frac{3}{5}\mathbf{y} - \frac{3}{5}\mathbf{x}$

(ii) $\mathbf{x} + \mathrm{k}\mathbf{y}$

(iii) $\mathrm{h} = \frac{5}{2}, \mathrm{k} = \frac{3}{2}$

REFLECTION

- Translation vectors create the largest part of this concept
- Other concepts such as the Cartesian plane, simultaneous equations and transformation permeate into vectors

22.7 Summary

This unit has brought a broader picture of the relationships that exist between directions and lengths (magnitudes) in real life. You are therefore encouraged to look closely into the assessment section so that you develop your vectors well.

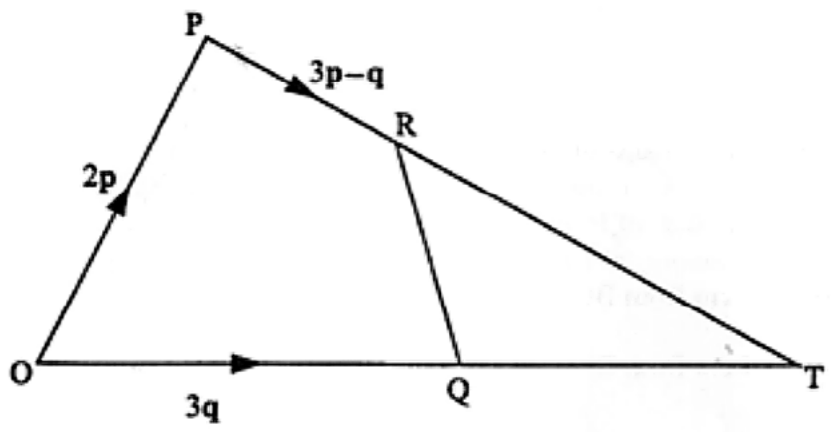
22.8 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

22.9 Assessment Test

1. QN 5(b) ZIMSEC NOV 2006 P2

(b)



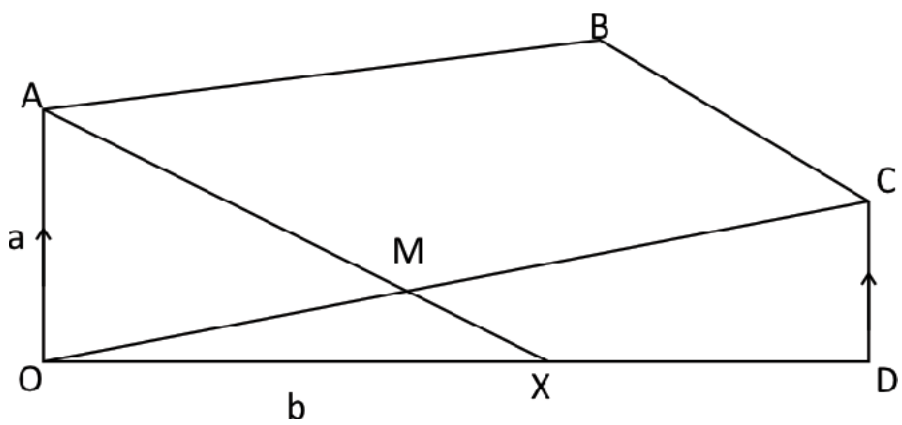
In the diagram, PRT and OQT are straight lines. $\overrightarrow{OP} = 2\mathbf{p}$, $\overrightarrow{OQ} = 3\mathbf{q}$ and $\overrightarrow{PR} = 3\mathbf{p} - \mathbf{q}$.

(i) Express $\overrightarrow{RQ}$ as simply as possible in terms of **p** and/or **q**. [2]

(ii) Given that PT = mPR, express $\overrightarrow{PT}$ in terms of **p**, **q** and m. [1]

(iii) Given also that OT = nOQ form an equation connecting **p**, **q**, m and n. Hence find the value of m and the value of n. [4]

2. Question 10 ZIMSEC NOV 2012 P2



OABCD is a pentagon such that AB is parallel to OD and DC is parallel to OA. M is the mid-point of OC such that AM produced cuts OD at X.

$\overrightarrow{OA} = \mathbf{a}$ $\overrightarrow{OD} = \mathbf{b}$ and $\overrightarrow{OD} = 3\overrightarrow{XD}$.

(a) Express the following in terms of **a** *and/or* **b**,

(i) AD,

(ii) $\overrightarrow{OX}$,

(iii) $\overrightarrow{AX}$. [3]

(b) If $\overrightarrow{MX} = k\overrightarrow{AX}$, express $\overrightarrow{MX}$ in terms of **a**, **b** *and* k. [1]

(c) Given that $\overrightarrow{DC} = h\overrightarrow{OA}$, express in terms of **a**, **b** *and* h,

(i) $\overrightarrow{OM}$,

(ii) $\overrightarrow{MX}$. [3]

(d) Using your results in **(b)** and **c(ii)**, find the value of h and the value of k. [3]

(e) Using your values of h and k in **(d)**, express in terms of **a** and/or **b**,

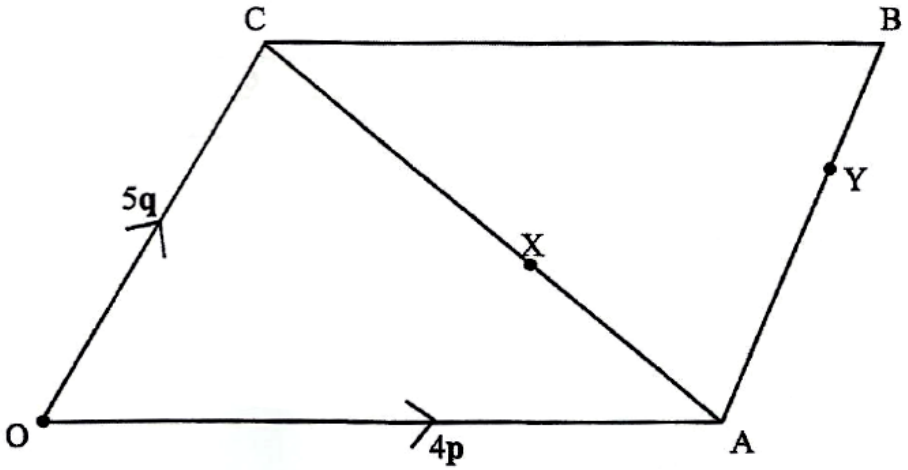
(i) $\overrightarrow{MX}$,

(ii) $\overrightarrow{DC}$. [2]

3. Question 8 ZIMSEC NOV 2013 P2

(a) If $\mathbf{g} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$ and $\mathbf{h} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, express $\mathbf{g} + 2\mathbf{h}$ in the form $\begin{pmatrix} x \\ y \end{pmatrix}$. [2]

(b)



In the diagram, OABC is a parallelogram in which $\overrightarrow{OA} = 4\mathbf{p}$ and $\overrightarrow{OC} = 5\mathbf{q}$. X is a point on AC such that AX:XC = 2:3.

(i) Express in terms of **p** and/or **q**

1. $\overrightarrow{AC}$,

2. $\overrightarrow{OX}$ in its simplest terms.

(ii) Y is a point on AB such that $\frac{AY}{AB} = k$, where k is a constant.

Express $\overrightarrow{OY}$ in terms of **p**, **q** and k.

(iii) Given that OY = hOX, where h is a constant,

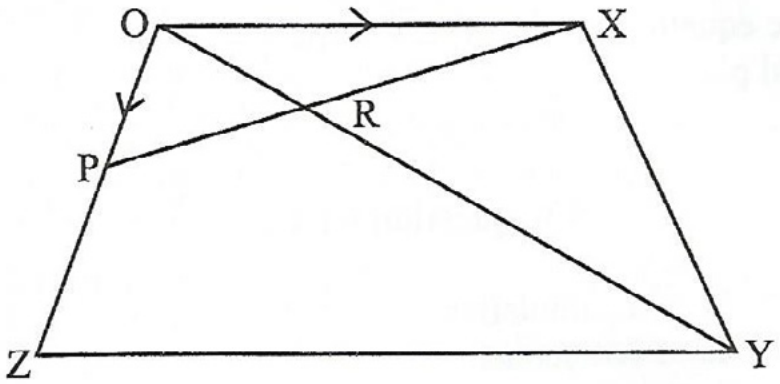
write down another expression for $\overrightarrow{OY}$ in terms of **p**, **q** and h.

(iv) Using results in **(ii)** and **(iii)**, find the value of h and the value of k.

(v) Express $\frac{\text{the area of } \Delta OAY}{\text{the area of parallelogram OABC}}$, as a fraction in its simplest form.

[10]

4. Question 11 ZIMSEC JUNE 2014 P2



In the diagram, OXYZ is a quadrilateral in which P is a point on OZ such that $\overrightarrow{OP} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ and $\overrightarrow{OX} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$. OY and XP intersect at R.

(a) Find $\overrightarrow{XP}$. [2]

(b) Given that $\overrightarrow{XR} = h\overrightarrow{XP}$,

(i) express $\overrightarrow{XR}$ in terms of h,

(ii) show that $\overrightarrow{OR} = \begin{pmatrix} 5-6h \\ -2h \end{pmatrix}$.

[3]

(c) Given also that $\overrightarrow{OZ} = 3\overrightarrow{OP}$ and $\overrightarrow{ZY} = 2\overrightarrow{OX}$, find $\overrightarrow{OY}$. [1]

(ii) show that $\overrightarrow{OR} = \begin{pmatrix} 5-6h \\ -2h \end{pmatrix}$.

[3]

(c) Given also that $\overrightarrow{OZ} = 3\overrightarrow{OP}$ and $\overrightarrow{ZY} = 2\overrightarrow{OX}$, find $\overrightarrow{OY}$. [1]

(e) Write down the numerical value of the ratio $\frac{XR}{RP}$. [1]

5. Question 10 ZIMSEC JUNE 2015 P2

(a) The point, M, has coordinates $(7; -3)$ and $\overrightarrow{RM} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$.

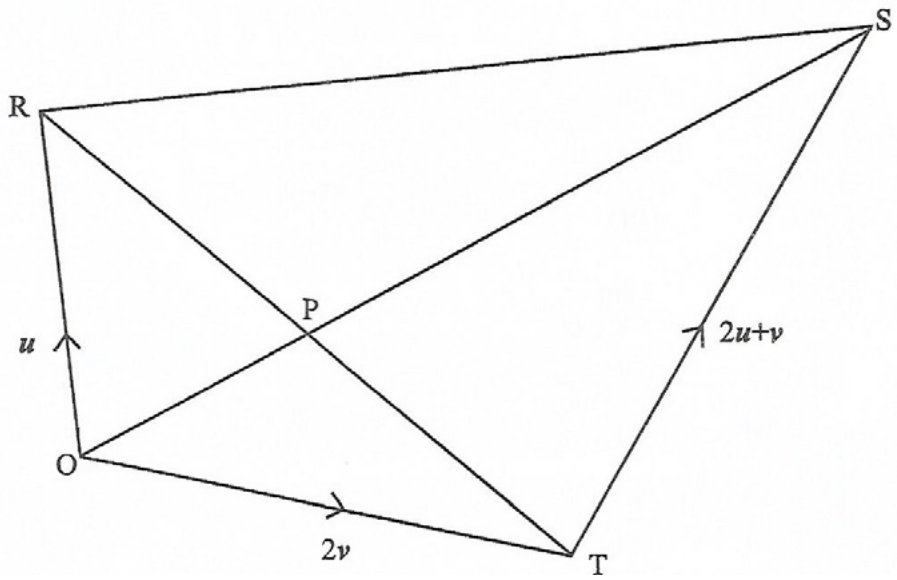
Calculate

(i) the coordinates of R,

(ii) $\overrightarrow{MR}$.

[3]

(b)



The diagram is a quadrilateral ORST in which $\overrightarrow{OR} = u$,

$\overrightarrow{OT} = 2v$ and $\overrightarrow{TS} = 2u + v$.

Diagonals OS and RT intersect at P.

(i) Express in terms of u and/or v.

1. $\overrightarrow{RT}$,

2. $\overrightarrow{OS}$.

(ii) Given that $\overrightarrow{OP} = k\overrightarrow{OS}$, express in terms of k, u and/or v

1. $\overrightarrow{OP}$,

2. $\overrightarrow{RP}$ and show that it reduces to $(2k-1)u+3kv$.

(iii) Given also that $\overrightarrow{RP} = h\overrightarrow{RT}$, express $\overrightarrow{RP}$ in terms of h, u and/or v.

(iv) Using the results in (ii) 2 and (iii), calculate the value of h and the value of k.

[9]

UNIT 23 - TRANSFORMATION 1

CONTENTS

23.1 INTRODUCTION

In this unit we are going to look into the basic introduction of transformation. We are simply going to consider four components of transformation, that is, translation, reflection, rotation and enlargement, that is, isometric transformation and non- isometric transformations. This unit will unpack these aspects in detail in the form of examples and illustrations as well as activities.

OBJECTIVES

After going through this unit, you should be able to

- define translation and reflection transformation
- translate plane figures
- reflect a point or plane figure in a given mirror line

KEY TERMS

Isometric: – the original shape maintains its dimensions (shape remains congruent to the original shape after transformation).

Non-Isometric: – the original shape is deformed, enlarged or reduced, that is it changes shape after transformation.

TIME: You are expected not to spend more than 8 hours on this unit.

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems on Transformation as possible for you to grasp all the concepts in this topic.

23.2 TRANSLATION

This type of a transformation is isometric since the shape remains congruent after transformation. Under this type of a transformation objects are moved from one point to another in a straight line and without turning or twisting. Vector notation can be used to represent translation vectors.

Figure 23.1 shows examples of translation vectors in the x – y plane. Note that A^{I} is a translated image of A, B^{I} is the translated image of B and so on.

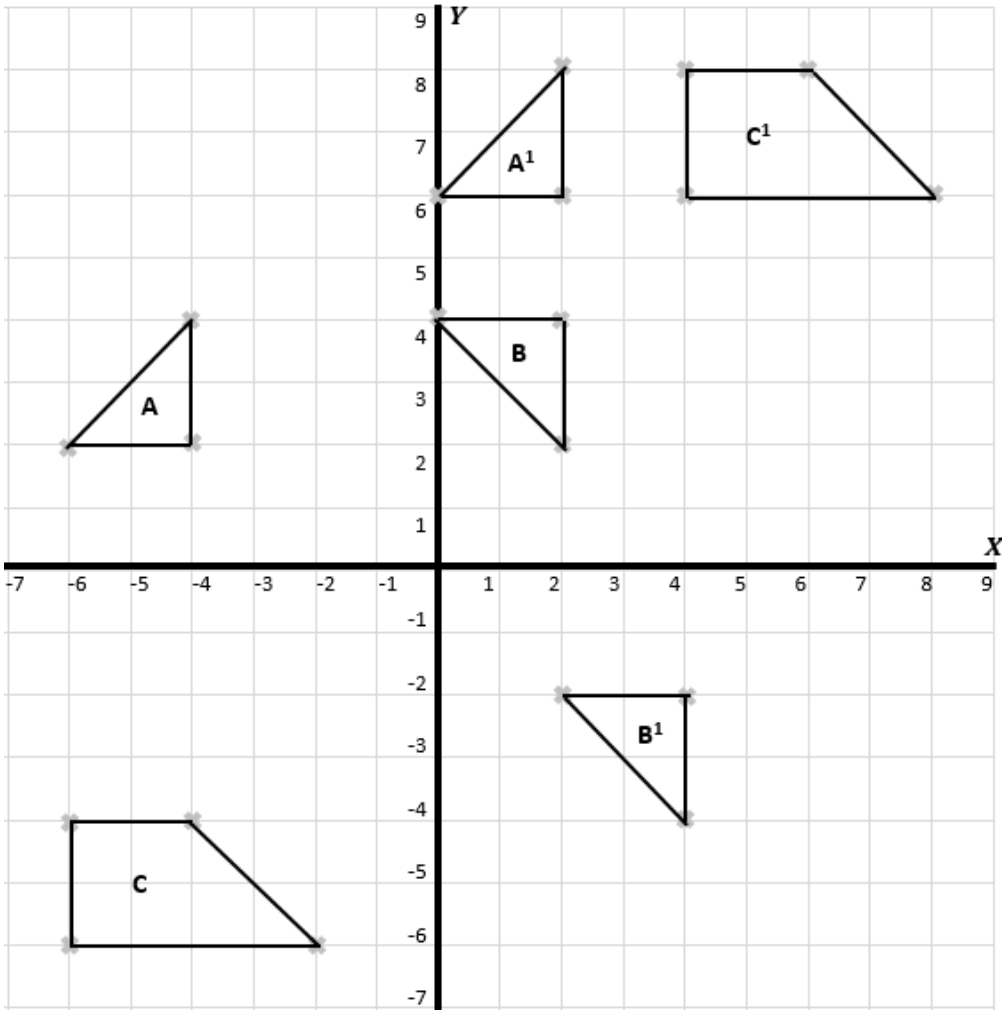


Figure 23.1

23.2.1 Vector notation of a translation vector

If a vector is translated in the x – y plane then the movements along the x-axis and y-axis are represented in column form of a vector, that is $\binom{x}{y}$.

Figure 23.2 below shows a translation vector and its column vector representation. Note that A^{I} is a translated image of A, B^{I} is the translated image of B and so on.

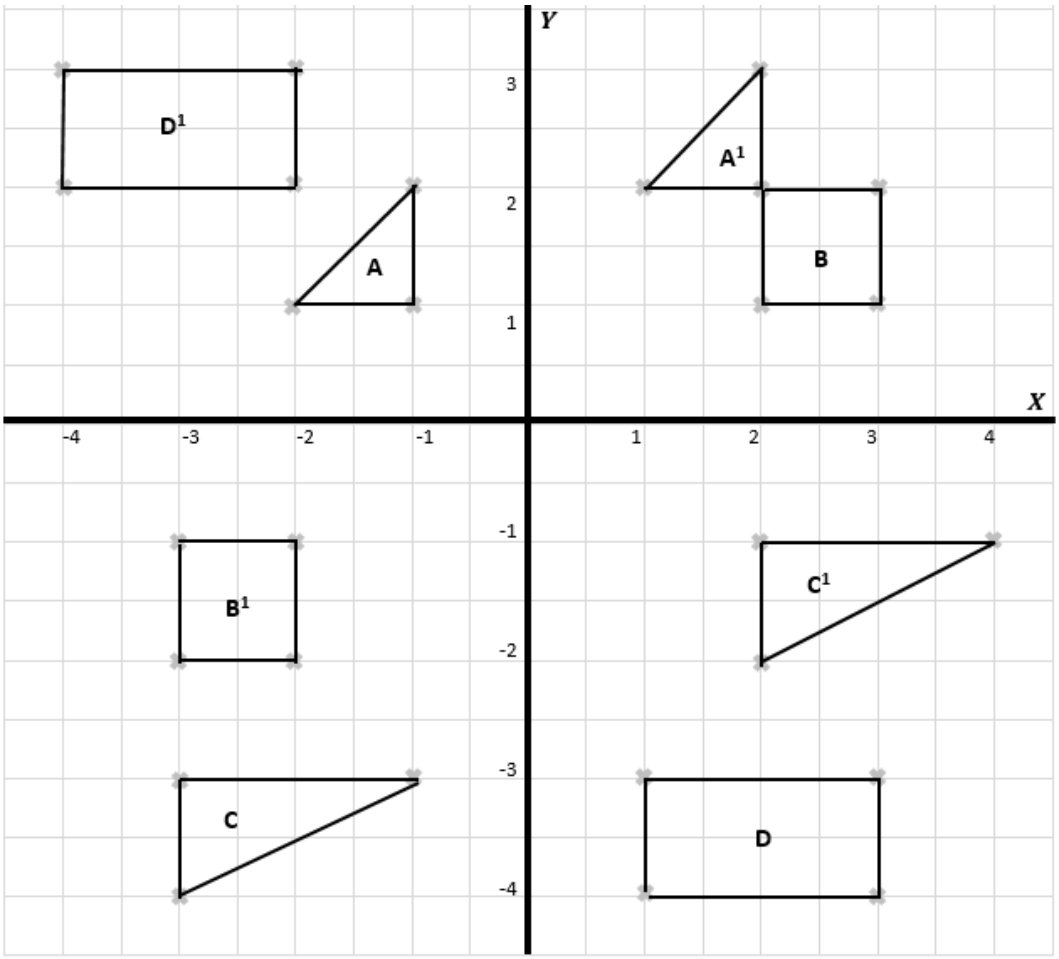


Figure 23.2

$\overrightarrow{AA^{I}} = \binom{3}{1}$, $\overrightarrow{BB^{I}} = \binom{-5}{-3}$, $\overrightarrow{CC^{I}} = \binom{5}{2}$ and $\overrightarrow{DD^{I}} = \binom{-5}{6}$

Activity (*23.1*) Translation transformation

Questions

1. In the diagrams below identify congruent shapes and state the column vector of each translation given

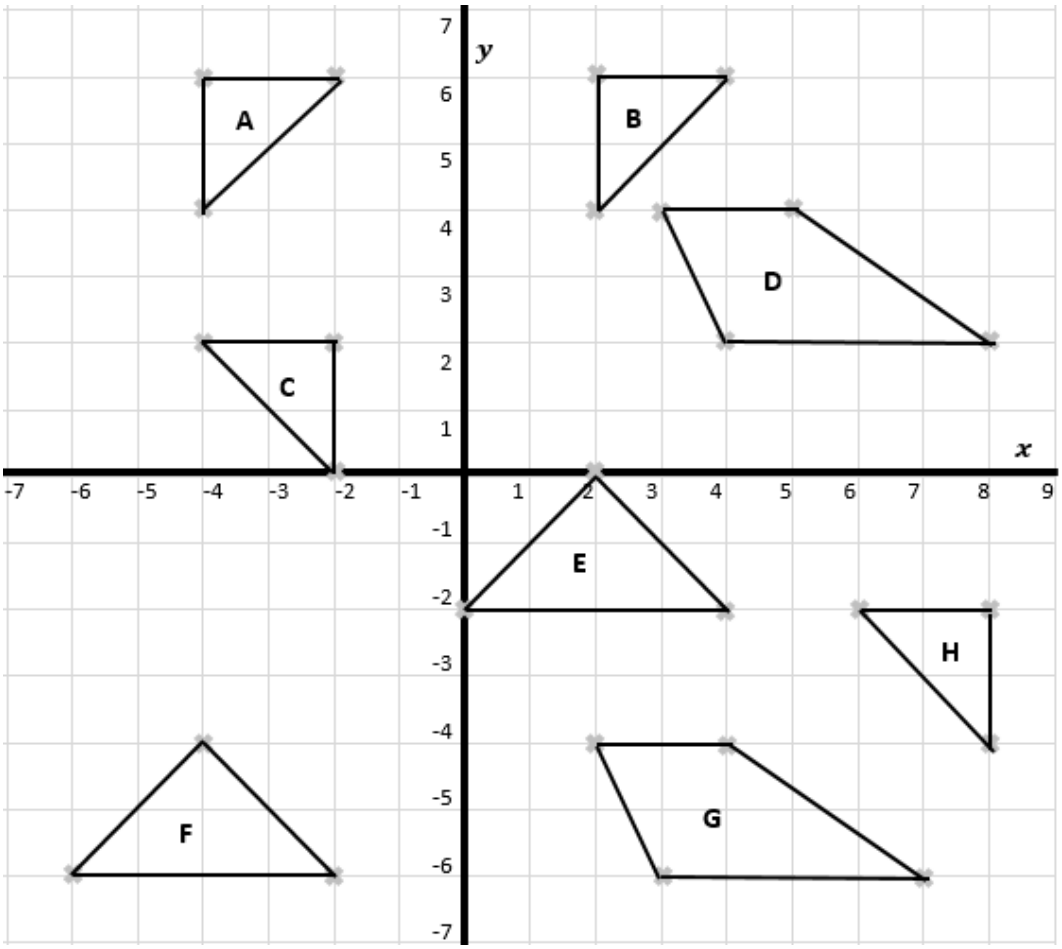


Figure 23.3

2. Find the image of the points given below under the transformations described by the given column vectors

(a) $(3;5), \begin{pmatrix}1\\2\end{pmatrix}$ (b) $(3;7), \begin{pmatrix}-3\\1\end{pmatrix}$ (c) $(-3;2), \begin{pmatrix}6\\3\end{pmatrix}$

(d) $(4;-1), \begin{pmatrix}5\\12\end{pmatrix}$ (e) $(-7;-5), \begin{pmatrix}2\\1\end{pmatrix}$ (f) $(-8;4), \begin{pmatrix}11\\-2\end{pmatrix}$

(g) $(13;15), \begin{pmatrix}8\\7\end{pmatrix}$ (h) $(9;1), \begin{pmatrix}0\\2\end{pmatrix}$ (i) $(1;1), \begin{pmatrix}-1\\-1\end{pmatrix}$

(j) $(-1;5), \begin{pmatrix}1\\2\end{pmatrix}$ (k) $(10;10), \begin{pmatrix}10\\10\end{pmatrix}$

3. Draw axes for x and y from – 3 to 10. Draw ΔXYZ with X (- 1; 5), Y (1;4) and Z (2; 5). Translate ΔXYZ using the vector $\binom{2}{1}$. Label this image $X_IY_IZ_1$ then translate $X_IY_IZ_1$ using vector $\binom{-1}{-1}$. Label this image $X_2Y_2Z_2$.

   Give the vectors describing the translation which map

   (a) $\Delta X_2Y_2Z_2$ to ΔXYZ (b) $\Delta X_2Y_2Z_2$ to $\Delta X_1Y_1Z_1$

23.3 ROTATION

This type of a transformation is isometric as well since the shapes remain congruent after transformation. Under this type of a transformation objects, shapes, points or lines are turned clockwise and anti-clockwise about a fixed point called a centre of rotation.

NOTE: Consider the following when describing a rotation

(a) State the name of the transformation, that is Rotation

(b) Centre of rotation in the form (x; y)

(c) The angle of rotation and direction (clockwise or anti-clockwise)

(d) The matrix of rotation (note that the use of matrices will be done in unit 25)

Worked Example [1]

Questions

Draw axes for x and y from – 4 to 4. Draw triangle ABC with A(0;1), B(2;3) and C(2;1). Draw the images of triangle ABC after clockwise rotation about the origin O

a) 90^0 rotation b) 180^0 rotation c) 270^0 rotation

Solutions

23.3.1 90° rotation

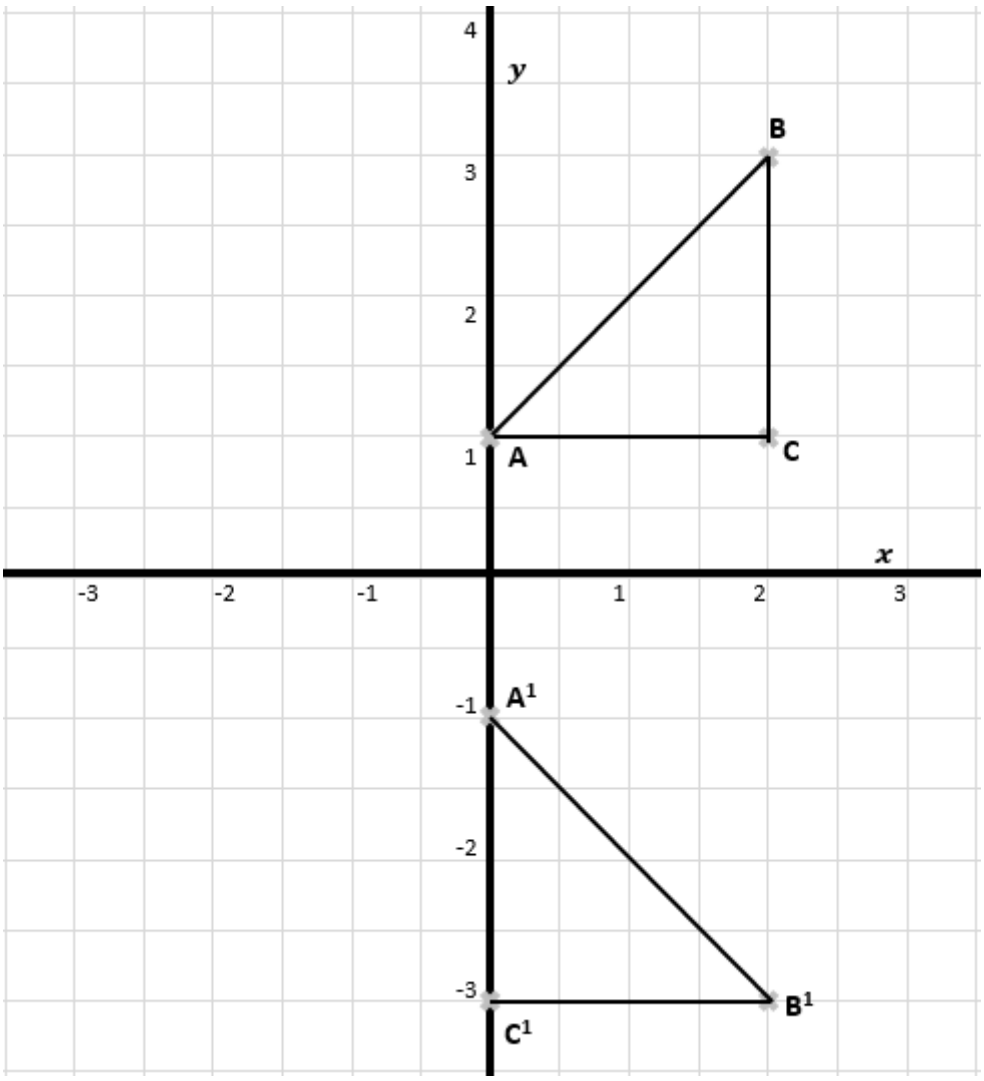


Fig 23.4

23.3.2 180° rotation

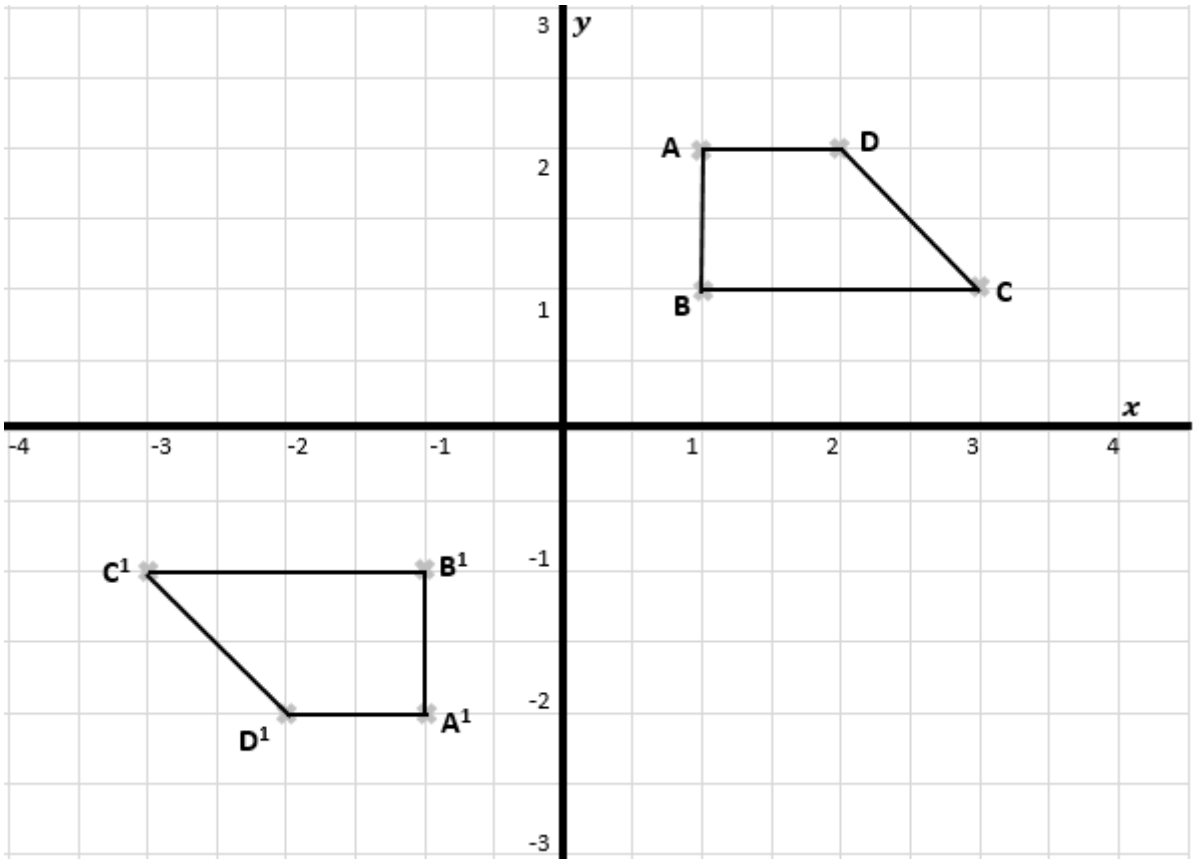


Fig 23.5

23.3.3 270° rotation

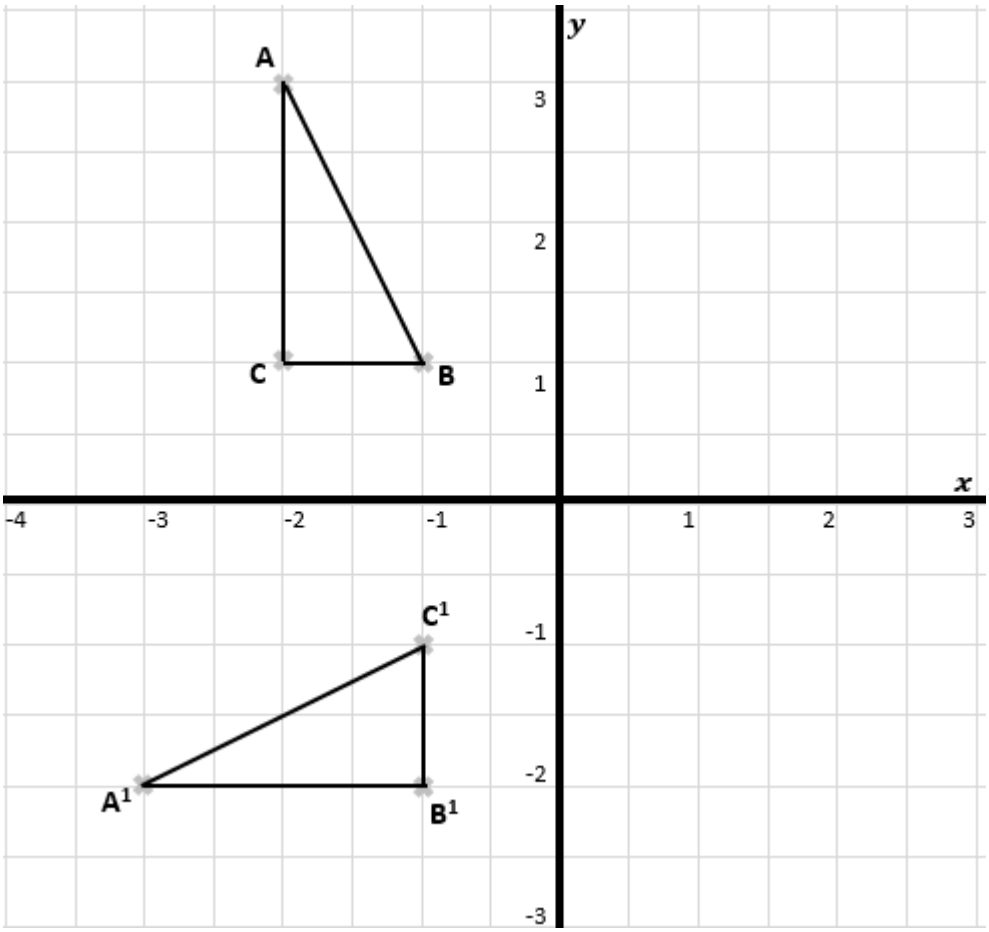


Fig 23.6

Worked Example [2]

Questions

Draw axes for x and y from – 3 to 3. Draw triangle ABC with A(1;1), B(2;1) and C(1;2). Draw the images of triangle ABC after anti-clockwise rotation about the origin O

a) 90^0 rotation b) 180^0 rotation c) 270^0 rotation

Solutions

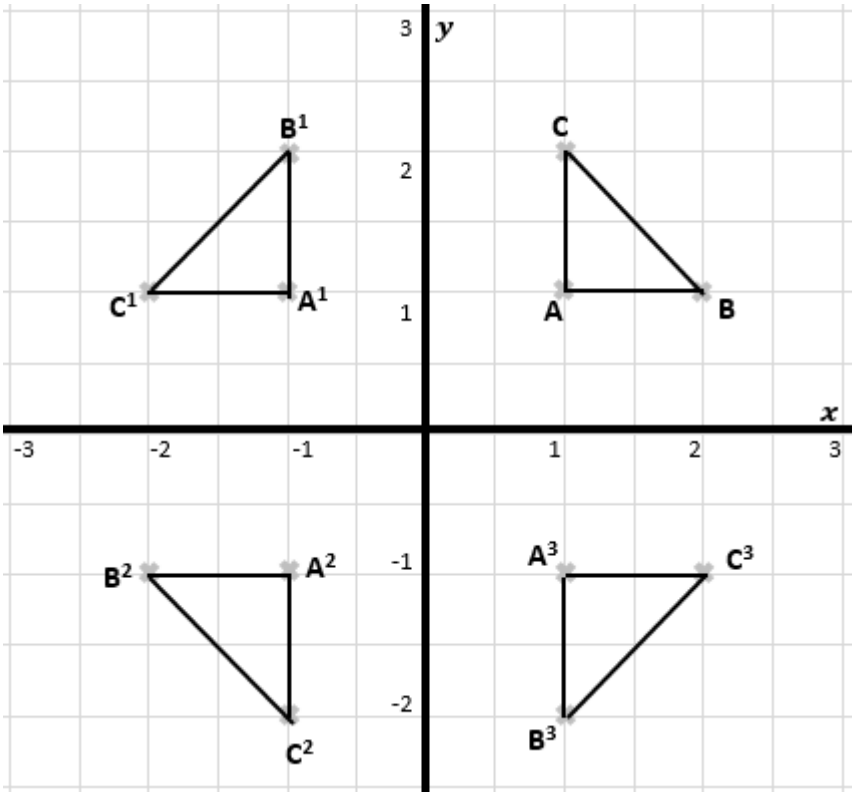


Fig 23.7

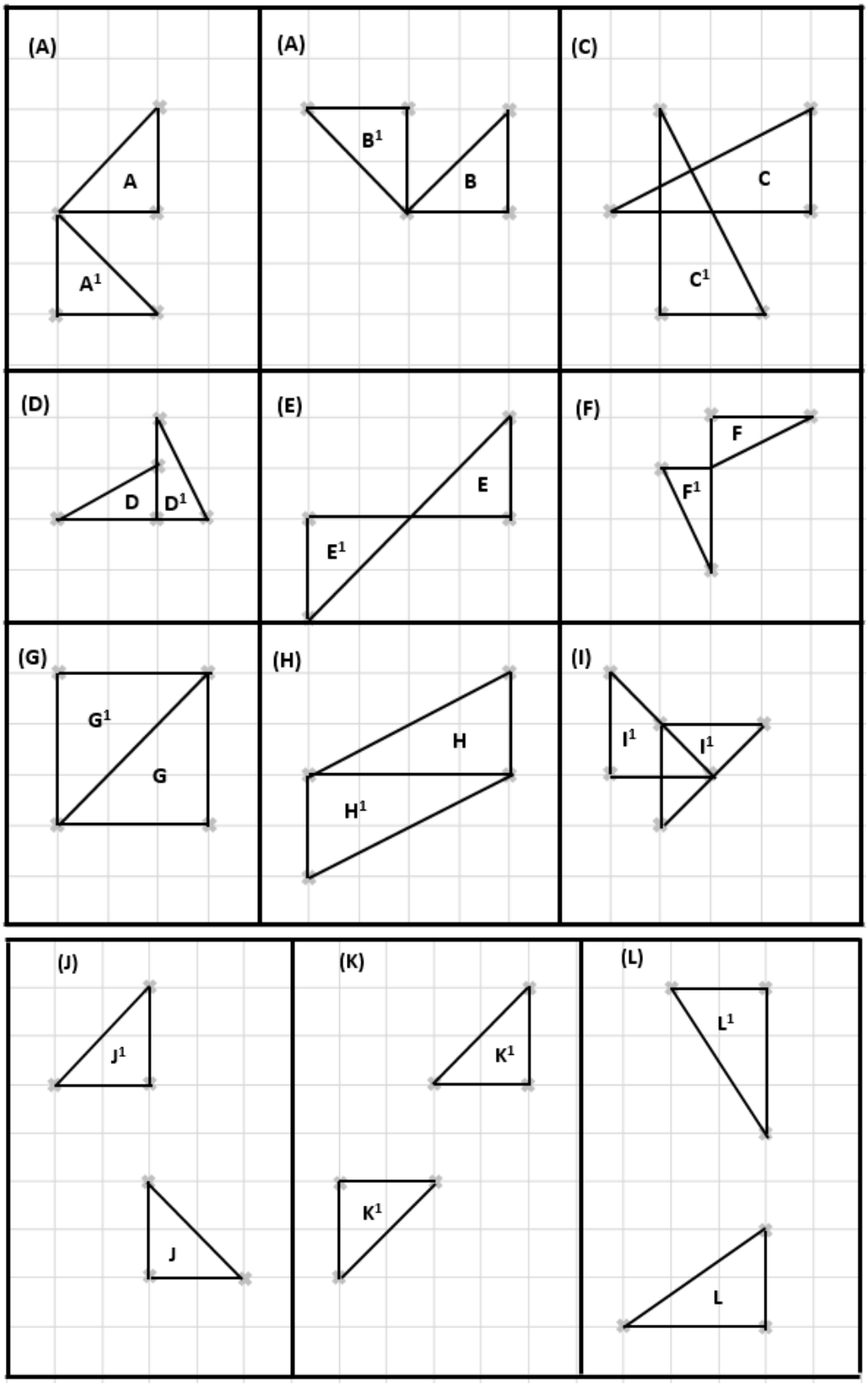
Note:

- a rotation of 90° clockwise is equivalent to a rotation of 270° anti-clockwise and vice-versa.
- a rotation of 180° clockwise is equivalent to 180° anti-clockwise, therefore there is no need to mention direction for rotation of 180°.

Activity (*23.2*) Rotation

Questions

1. In the diagrams below, give the angle of rotation when $\Delta A, \Delta B, \Delta C, \Delta D, \ldots \Delta L$ is mapped to $\Delta A^{I}, \Delta B^{I}, \Delta C^{I}, \Delta D^{I}, \ldots \Delta L^{I}$, as shown in figure 23.8 below
2.



23.4 REFLECTION

Again this type of a transformation is isometric since the shapes remain congruent after transformation. Under this type of a transformation objects, shapes, points or lines are reflected on a mirror line which can be the x-axis, y-axis, y , y = - x or any other line

Consider the following when describing a reflection

(a) State the name of the transformation, that is reflection.

(b) State the reflection line ($y = 0, x = 0, y = x, y = -x$ or $y = mx + c$) where possible.

Tips

- The shape, object, point or line under the reflection is the same with its image.
- The mirror line lies centrally in-between the shape, object, point or line under the reflection, that makes the image perpendicular to its pre-image.
- The axis of reflection is an invariant line.

In the diagram below is a original object and its image showing a reflection line and all points being perpendicular. This activity can also be done practically using a mirror.

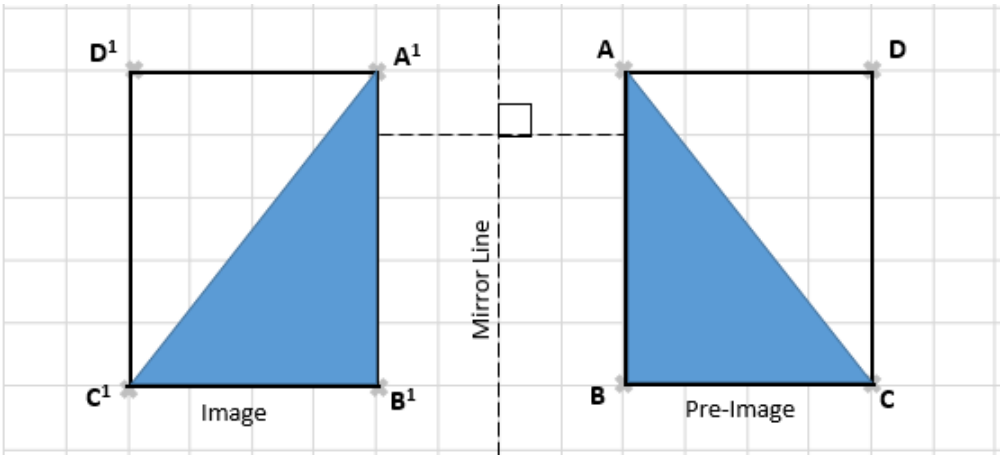
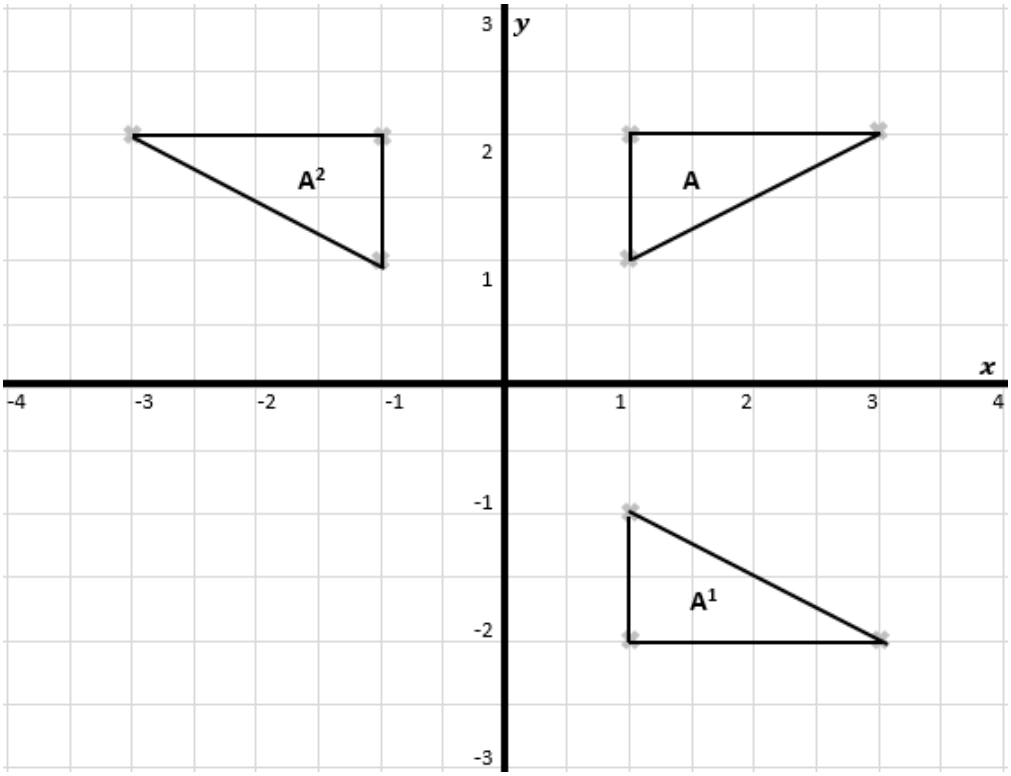


Figure 23.9

Worked Example [3]

Question



The diagram above shows triangle A being mapped to A_1 and A_2. State the transformation and the mirror lines

Solution

Triangle A is mapped onto A_1 by a reflection transformation and the mirror line is $y = 0$ (x – axis).

Triangle A is mapped onto A_2 by a reflection transformation and the mirror line is $x = 0$ (y – axis).

Hints:

- The mirror line lies centrally in-between the shape or object. The mirror line is a line that makes the image perpendicular to its pre-image.
- The mirror line can be drawn by simply connecting corresponding points of the original object and its image then find the line that lies centrally and perpendicular to the points.

Worked Example [4]

Question

In the diagram below ΔABC being mapped onto $\Delta A^1B^1C^1$ by a reflection along the line $y = -x - 1$ as shown. Identify the mirror line.

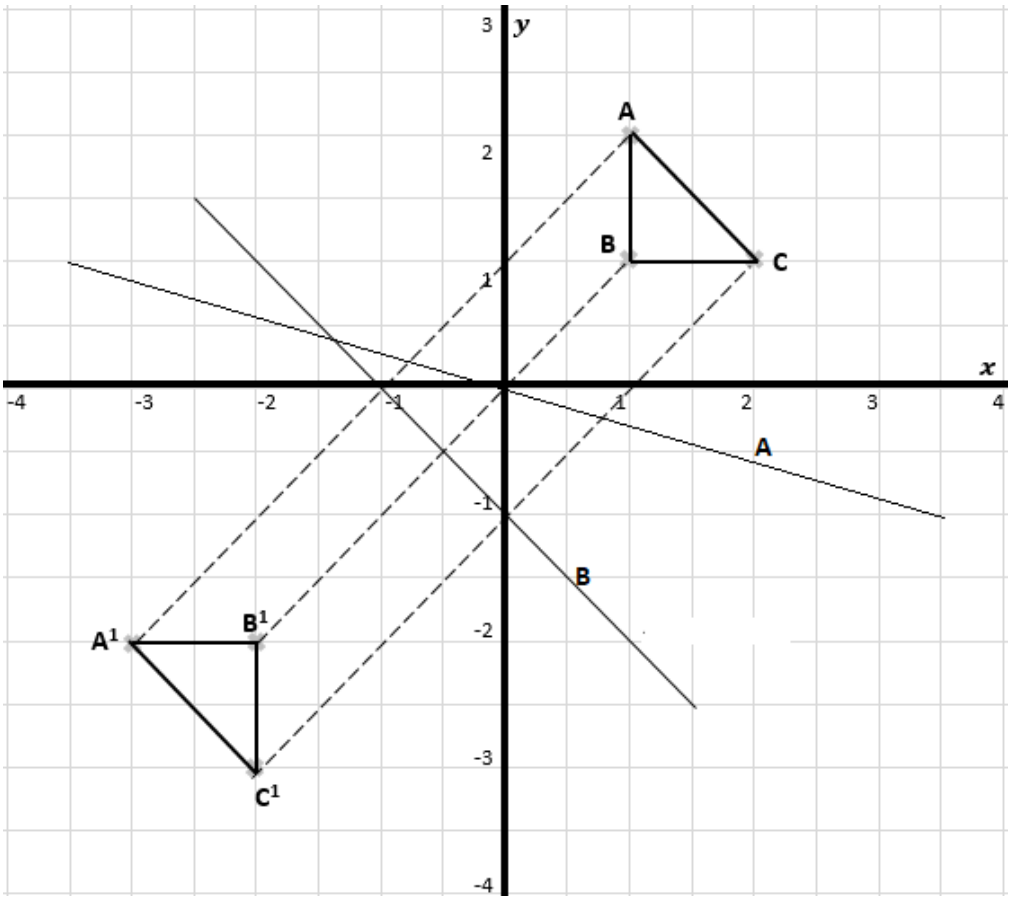


Figure 23.11

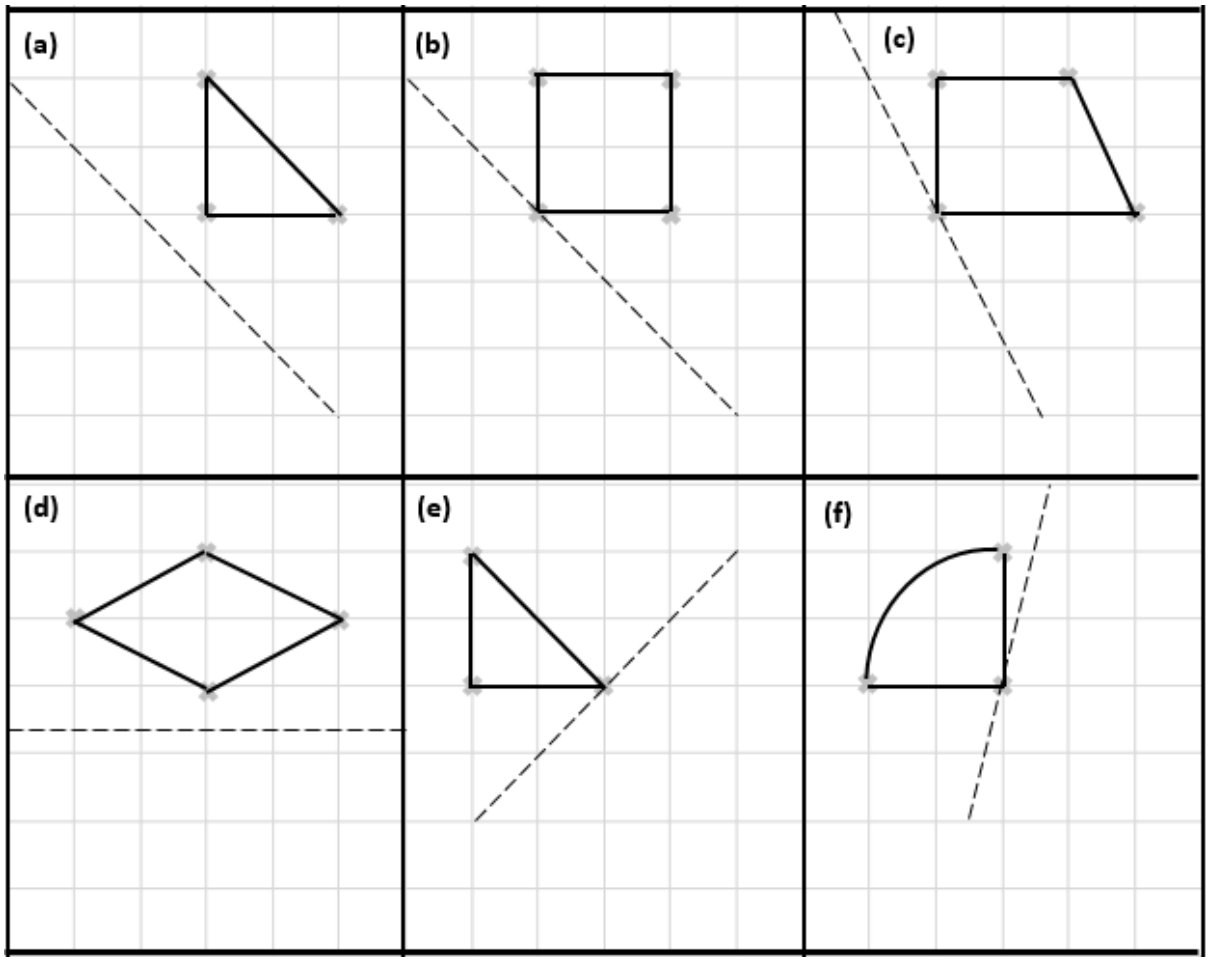
Solution

The mirror line is line B.

Activity (*23.3*) Reflection

Questions

1. Draw the image of each shape under the reflection line given in broken lines.



2. Use a scale of 2 cm to represent 2 units on both axes

   (a) Draw axes, for x from – 6 to 5 and for y from 0 to 5. Draw ΔXYZ by plotting X(1;2), Y(3;2) and Z(3;5). Draw the image $\Delta X_1Y_1Z_1$ when ΔXYZ is reflected in the y – axis.

   (b) Draw axes, for x from – 1 to 5 and for y from – 3 to 4. Draw ΔABC by plotting A(1;-1), B(5;-1) and C(4;0). Draw the image $\Delta A_1B_1C_1$ when ΔABC is reflected in the x – axis.

   (c) Draw axes, for x from – 7 to 2 and for y from – 5 to 1. Draw $\Delta ABCD$ by plotting A(-3;-1), B(-3;-2), C(-5;-2) and D(-5;-1). Draw the reflection line y = x. Draw the image $\Delta A_1B_1C_1D_1$ when $\Delta ABCD$ is reflected in the mirror line y = x.

(d) Draw axes for x and y from – 6 to 9. Draw ΔPQR where P is (-6;-2), Q is (-3;-4) and R is (-2;-1). Draw the following images of ΔPQR

(i) $\Delta P_1Q_1R_1$ by reflection in the x – axis

(ii) $\Delta P_2Q_2R_2$ by reflection in the line y = x

(iii) $\Delta P_3Q_3R_3$ by reflection in the line y = - x

(iv) $\Delta P_4Q_4R_4$ by reflection in the line y – axis

(v) $\Delta P_5Q_5R_5$ by reflection in the line x = 1

23.5 ENLARGEMENT

This type of a transformation is non-isometric since the shape changes in size after transformation. Under this type of a transformation objects are increased or decreased in size but the objects remain similar after transformation.

Think of a picture thrown on the screen when a slide projector is used. The diagram below shows such type of an enlargement

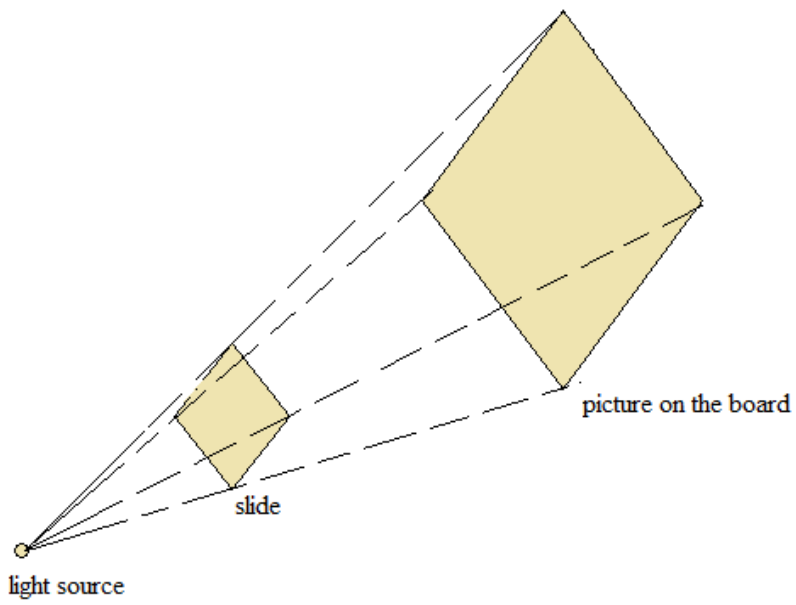


Figure 23.12

Note: Consider the following when describing a reflection

(a) State the name of the transformation, that is enlargement

(b) State the centre of enlargement

(c) State the scale factor of enlargement

Tips

- The centre of enlargement is invariant under enlargement
- The **centre of enlargement** can be found by connecting corresponding points of the original object and the image object (geometrical way)
- The scale factor is **negative** when the enlarged object is on the opposite side of the centre.

23.5.1 Positive enlargement scale factor

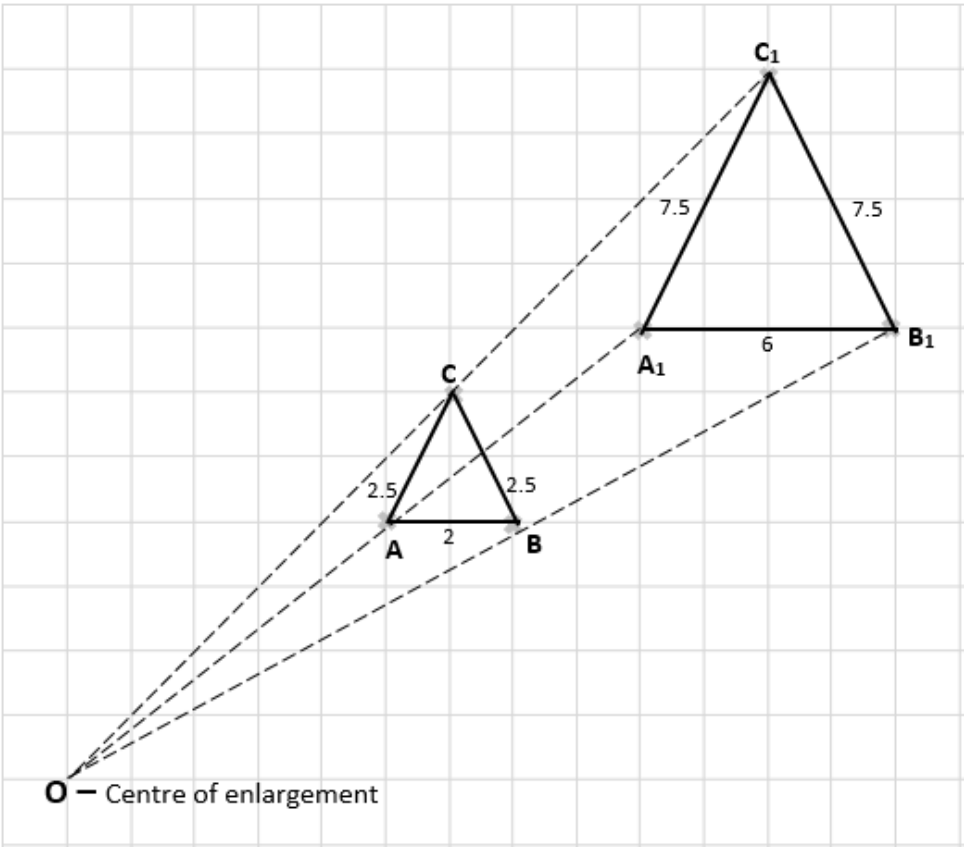


Fig 23.13

By simply dividing corresponding dimensions we can find the scale factor of enlargement of ΔABC onto $\Delta A_1B_1C_1$,

that is $\frac{A_1B_1}{AB} = \frac{6}{2} = +3$ **or** $\frac{A_1C_1}{AC} = \frac{7.5}{2.5} = +3$ and so on

23.5.2 Negative enlargement scale factor

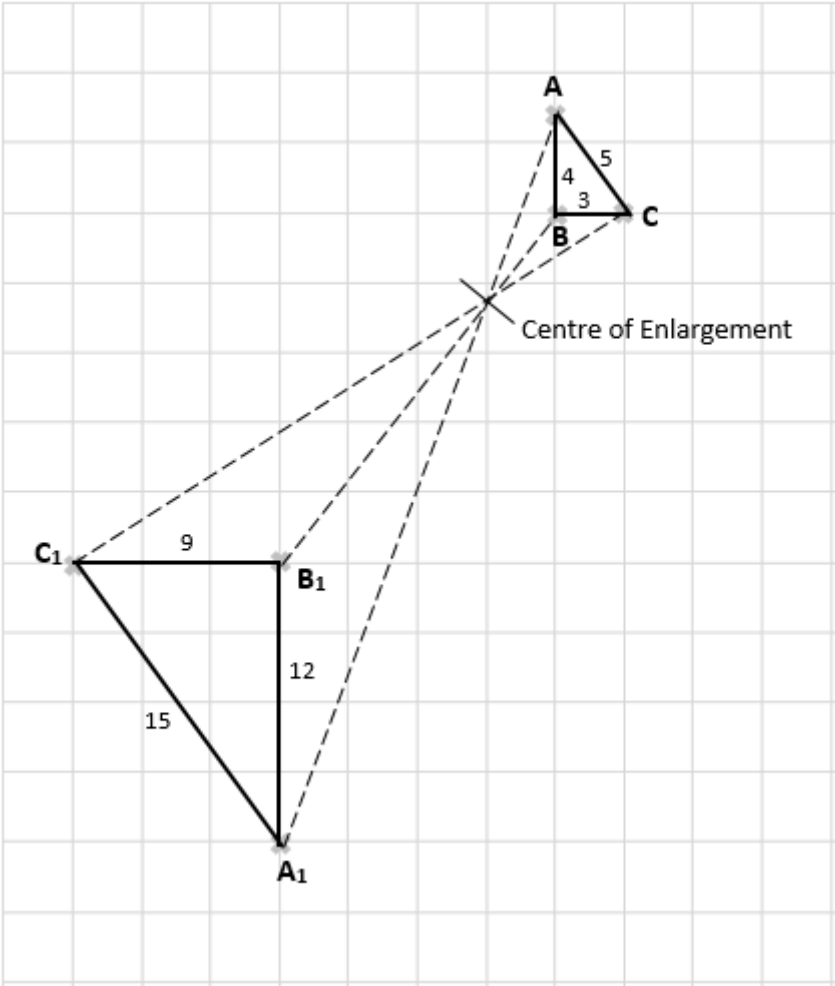


Figure 23.14

By simply dividing corresponding dimensions we can find the scale factor of enlargement of ΔABC onto $\Delta A_1B_1C_1$, that is $\frac{A_1B_1}{AB} = \frac{-12}{4} = -3$ or $\frac{A_1C_1}{AC} = \frac{-15}{5} = -3$ and so on.

23.5.3 Fractional scale factor

We can reverse the enlargement process and reduce or shrink the object.

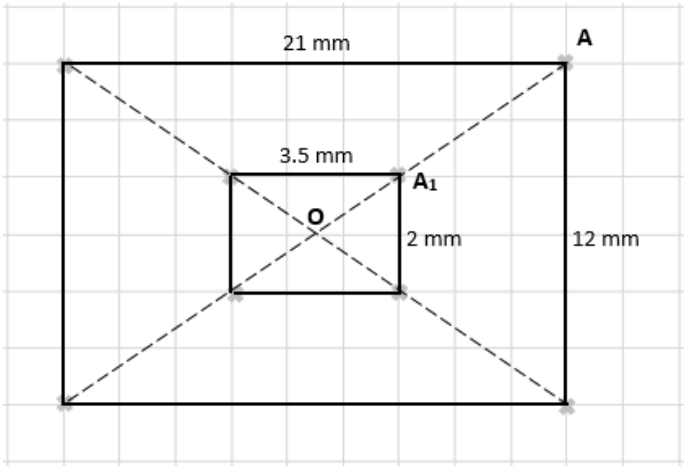


Fig 23.15

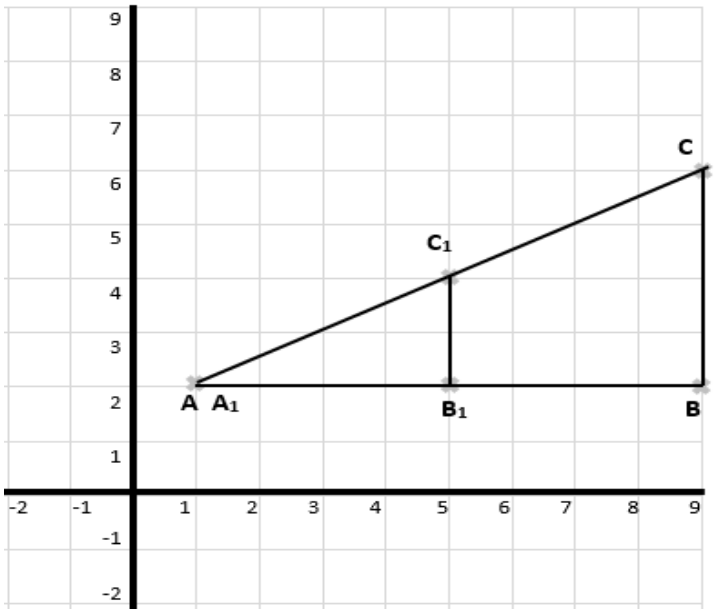
The scale factor of enlargement in figure 23.15 is $\frac{1}{6}$,where rectangle A is mapped onto rectangle A_1 and O is the centre of enlargement

Worked Example [5]

Questions

Draw axes for x and y from 0 to 9 using 2cm as 1 unit. Draw ΔABC with A(1;2), B(9;2) and C(9;6) and $\Delta A_1B_1C_1$ with $A_1(1;2), B_1(5;2)$ and $C_1(5;4)$. State the centre of enlargement and the scale factor of enlargement.

Solution



A is the centre of enlargement

The scale factor is $\frac{A_1B_1}{AB} = \frac{8 \text{ units}}{4 \text{ units}} = 2$

Activity (*23.4*) Enlargement

Questions

1. On a Cartesian plane enlarge the objects, using the given centre of enlargement and the scale factor indicated in Table 23.1

Table 23.1

Object	Centre	Scale factor
(2;3), (-1;3), (3;1)	(1;1)	1 ½
(5;1), (2;2), (4;2)	(2;-3)	-1 ½
(1;3),(2;3), (2;4)	(0;1)	-2
(3;5), (5;3), (6;6)	(2;3)	2
(8;1), (12;2), (12;6)	(6;5)	-½

2. Draw axes for x and y -1 to 10 using a scale of 1cm to represent 1 unit on both axes. Draw ΔXYZ, where X(2;3), Y(4;1), Z(5;4)

   Draw $\Delta X_1Y_1Z_1$, where X_1(2;5), Y_1(6;1), Z_1(8;7)

   (a) Identify the centre of enlargement, label it O and write down the coordinates of the centre in the form $(x; y)$

   (b) Find the scale factor of enlargement

3. Draw axes for x and y from – 8 to 8. Draw triangle ABC with A(6;- 1), B(4; - 3) C(4; - 1) and triangle XYZ with X(– 3;2), Y(1;6) and Z(1;2). If ΔABC is mapped onto ΔXYZ by a single transformation, find
   (a) The centre of enlargement
   (b) The scale factor of enlargement

4. Draw axes for x and y from – 8 to 8. Draw triangle PQR with P(2;3), Q(4;3) and R(2;6). If the triangle PQR is enlarged by a scale factor – 3 at a centre (2;3), draw the image of triangle PQR and label it $P_1Q_1R_1$,

5. Draw axes for x and y from – 8 to 8. Draw square ABCD with A(1;1), B(5;1), C(5;-3) D(1;-3), and WXYZ with W(-2;2), X(– 4;2), Y(-4;4), Z(-2;4). If $ABCD$ is mapped onto $WXYZ$ by a single transformation, find
   (c) The centre of enlargement
   (d) The scale factor of enlargement

REFLECTION

- Translation, Reflection and Rotation and translation are isometric transformations.
- Enlargement is a non-isometric transformation.
- Transformation as a topic requires knowledge of other concepts, such as geometrical construction, vectors, use of the Cartesian plane

23.6 Summary

This unit has explored in detail the basic skills required in tackling translation, rotation, reflection and enlargement transformation problems. The concepts in reflection, rotation and enlargement will be expanded further in unit 25 where the use of matrices and more geometrical ways of solving will be introduced. However, the translation transformation has been dealt with completely since it is a concept carried forward from vectors in unit 22.

23.7 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

23.8 Assessment Test

June 2004 paper 2 QN 8

1. Triangle ABC has vertices at A(1; 2), B(3; 2) and C(2: 1).
   Using a scale of 2cm to 1 unit on each axis, draw the x and y-axes for $-5 \leq x \leq 4$ and $-4 \leq y \leq 5$
   (a) Draw and label triangle ABC [1]
   (b) Triangle ABC is mapped onto triangle $A_1B_1C_1$ by an enlargement of scale factor 2 with (2; -1)as centre. Draw and label triangle $A_1B_1C_1$ [2]

2. Triangle ABC has vertices A(2; 1), B(4; 1) and C(4; 4). Using a scale of 1cm to represent 1 unit on both axes, draw the x and y-axes for $-4 \leq x \leq 14$ and $-10 \leq y \leq 12$
   (a) Draw and label triangle ABC [1]
   (b) Triangle $A_1B_1C_1$ is a reflection of triangle ABC in the line $y = -2$. Draw and label clearly triangle $A_1B_1C_1$ [3]

3. Triangle A has vertices at (2; 2), (5; 2) and (8; 4). Using a scale of 2cm to represent 2 units on both axes, draw the x and y-axes for $-8 \leq x \leq 10$ and $-8 \leq y \leq 8$
   (a) Draw and label triangle A [1]

(b) Triangle A is mapped onto triangle B by a translation $\begin{pmatrix}-9\\2\end{pmatrix}$. Draw and label triangle B [1]

(c) Triangle A is reflected onto triangle C in the line $y = -x$. Draw and label triangle C [2]

4. Triangle W has vertices at (1; -1), (7, -1) and (4; 4). Using a scale of 2cm to represent 2 units on both axes. Draw the x and y-axes for $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.

(a) Draw and label triangle W [1]

(b) Triangle X is an image of triangle W under the reflection in the line $y = x + 2$. Draw and label clearly,

(i) the line $y = x + 2$

(ii) triangle X [3]

5. Use a scale of 2cm to represent 2 units on x-axis and 2cm to represent 1 unit for y-axis for $-4 \leq x \leq 14$ and $-5 \leq y \leq 5$.

(i) Triangle ABC has vertices A(4; 1), B(6; 1) and C(6;2). Draw and label triangle ABC

(ii) Transformation T represents a translation vector $\begin{pmatrix}-2\\-6\end{pmatrix}$.
Draw and label triangle $A_1B_1C_1$, the image of triangle ABC under T.

(iii) Transformation R represents a clockwise rotation of about (4; 4).
Draw and label triangle $A_2B_2C_2$, the image of triangle ABC under R

6.

Triangle ABC has vertices at A(1; 2), B(3; 2) and C(2; 1).

Using a scale of 2cm to represent 1 unit on each axis, draw the x and y axes for $-5 \leq x \leq 4$ and $-4 \leq y \leq 5$.

(a) Draw and label triangle ABC. [1]

(b) Triangle ABC is mapped onto triangle $A_1B_1C_1$ by an enlargement of scale factor 2 with (2; –1) as centre. Draw and label triangle $A_1B_1C_1$. [2]

7. June 2011 paper 2 QN 8

Answer the whole of this question on a sheet of graph paper.

Triangle ABC has vertices A(2; 1), B(4; 1) and C(4; 4). Using a scale of 1 cm to represent 1 unit on both axes, draw the x and y-axes for $-4 \le x \le 14$ and $-10 \le y \le 12$.

(a) Draw and label clearly triangle ABC. [1]

(b) Triangle $A_1B_1C_1$ is a reflection of triangle ABC in the line $y = -2$. Draw and label clearly triangle $A_1B_1C_1$. [3]

8. June 2013 paper 2 QN 9

Triangle A has vertices at (2; 2), (5; 2) and (8, 1). Using a scale of 2 cm to represent 2 units on both axes, draw the x and y axes for $-8 \le x \le 10$ and $-8 \le y \le 8$.

(a) Draw and label triangle A. [1]

(b) Triangle A is mapped onto triangle B by a translation $\begin{pmatrix} -9 \\ 2 \end{pmatrix}$. Draw and label triangle B. [1]

(c) Triangle A is reflected onto triangle C in the line $y = -x$. Draw and label triangle C. [2]

9. June 2014 paper 2 QN

Triangle W has vertices at $(1; -1)$, $(7; -1)$ and $(4; 4)$. Using a scale of 2 cm to represent 2 units on both axes, draw the x and y-axes for $-10 \le x \le 10$ and $-10 \le y \le 10$.

(a) Draw and label clearly triangle W. [1]

(b) Triangle X is the image of triangle W under a reflection in the line $y = x + 2$.

Draw and label clearly,

(i) the line $y = x + 2$,

(ii) triangle X.

[3]

10. Nov 2014 Paper 2 QN 10(b)

Use a scale of 2 cm to represent 2 units on x-axis and 2 cm to represent 1 unit on y-axis for $-4 \le x \le 14$ and $-5 \le y \le 5$.

(i) Triangle ABC has vertices A (4; 1), B (6; 1) and C (6; 2).

Draw and label triangle ABC.

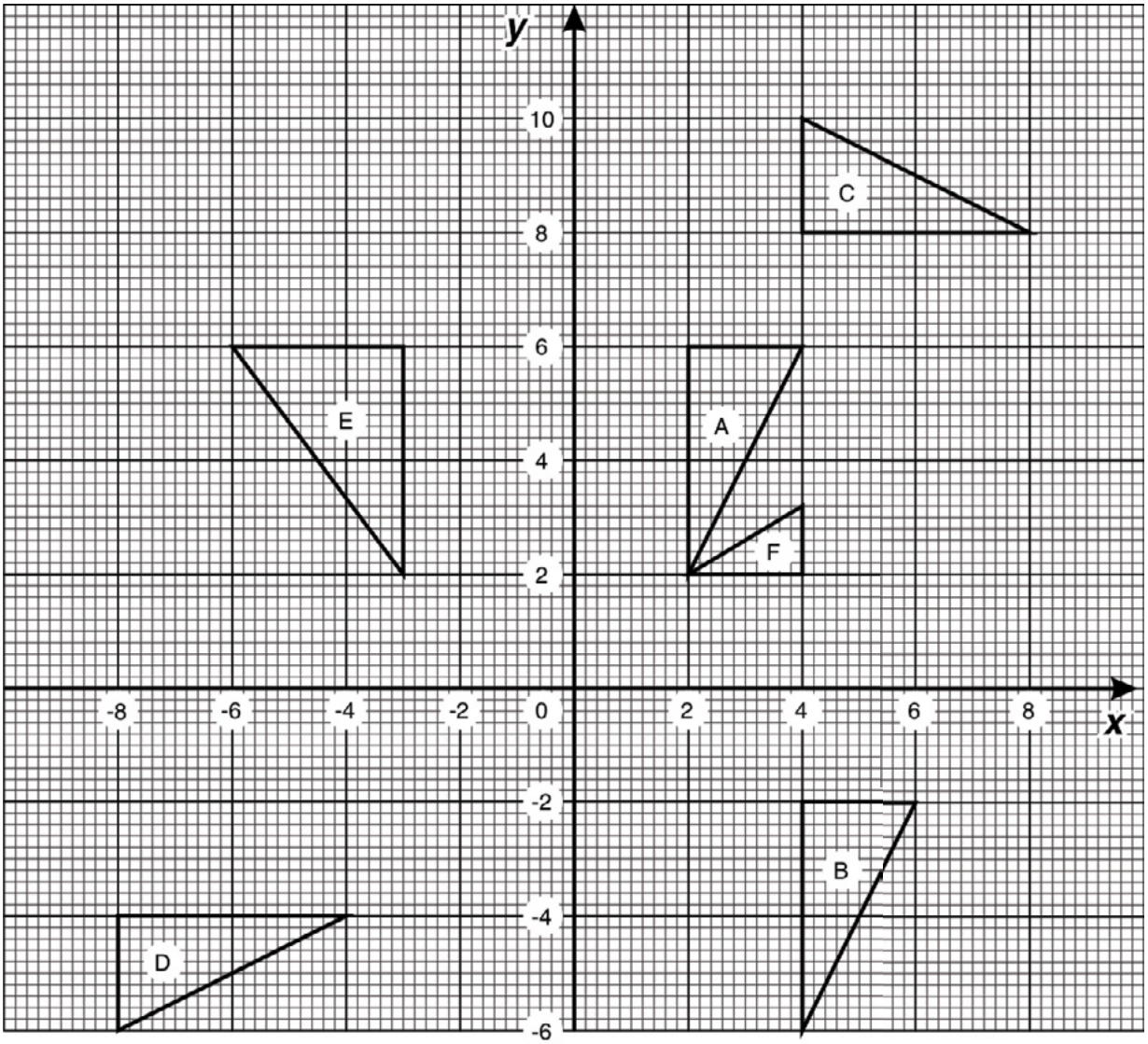
(ii) Transformation **T** represents a translation vector $\begin{pmatrix} -2 \\ -6 \end{pmatrix}$.

Draw and label triangle $A_1B_1C_1$, the image of triangle ABC under **T**.

(iii) Transformation **R** represents a clockwise rotation of 90° about (4; 4).

Draw and label triangle $A_2B_2C_2$, the image of triangle ABC under **R**.

11. Nov 2007 paper 2 QN 10



(i) ΔA is mapped onto ΔB by a translation. Write down the column vector for this translation. [1]

(ii) Describe **fully** the **single** transformation which maps ΔA onto ΔC. [3]

(iii) A reflection maps ΔA onto ΔD. Write down the equation of the mirror line. [2]

(iv) ΔD is mapped onto ΔF by a single transformation. Write down the matrix which represents this transformation. [2]

UNIT 24 - FUNCTIONAL GRAPHS 2

CONTENTS

24.1 INTRODUCTION

This unit is a continuation of Unit 5 on Functional Graphs. This unit goes further from Linear Graphs to Quadratic, Inverse and Cubic graphs.

Objectives

After going through the unit, you should be able to:

- ✓ use and interpret the functional notation
- ✓ construct a table of values from a given function
- ✓ use given scale correctly to construct the functions
- ✓ draw graphs (functions) of the form
  - $y = ax^2 + bx + c$ (quadratic function)
  - $y = \frac{1}{ax+b}$ (inverse function)
- ✓ estimate the area under the curve
- ✓ draw tangents at given points
- ✓ estimate the gradient of lines and curves at given points
- ✓ solve equations graphically
- ✓ use graphs to determine the maximum and minimum turning points

Key words

The following are key words and their meaning in Mathematics

Tangent – this is a straight line which passes through a point on a curve, it just touches a curve at a point

Gradient - this is the measure of the steepness of a line.

Line of symmetry – the line which divide a shape (a curve in our case) into two equal parts

Intersection- is the point where two lines or curves meet or join.

Maximum- this is the highest of greatest point reached

Minimum - this is the least value of the smallest value reached

Time

You are advised not to spend more than 10 hours in this unit.

Study skills

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems on Functional Graphs as possible for you to grasp all the concepts in this topic.

24.2 QUADRATIC FUNCTION/ GRAPHS

What is a quadratic function? A quadratic function or graph is a function with the highest power of the variable being 2. Below are examples of quadratic functions

24.2.1 Examples of Quadratic Functions

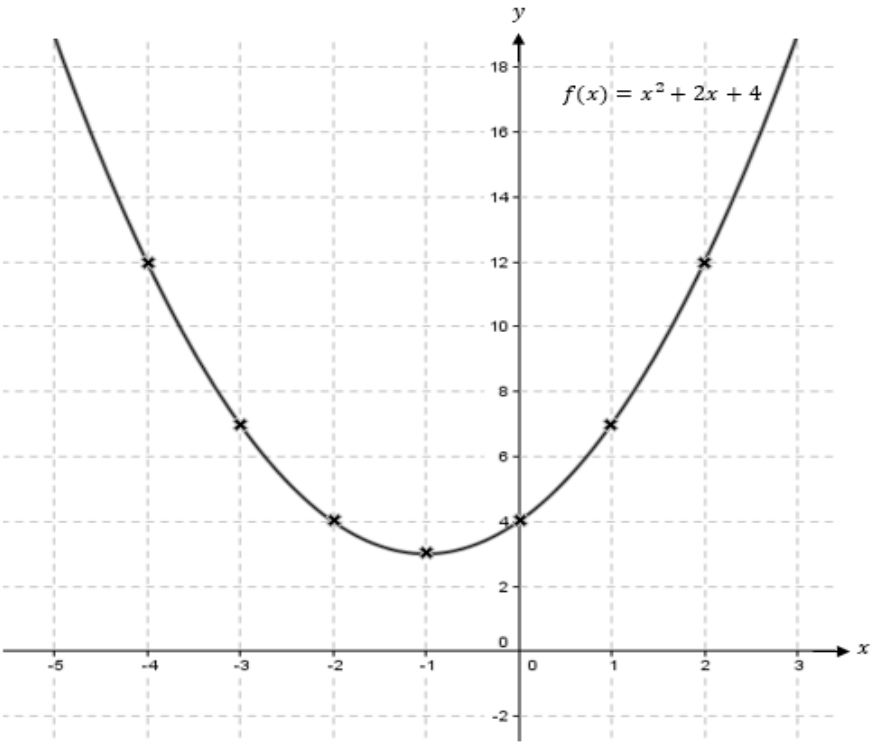


Fig 24.1

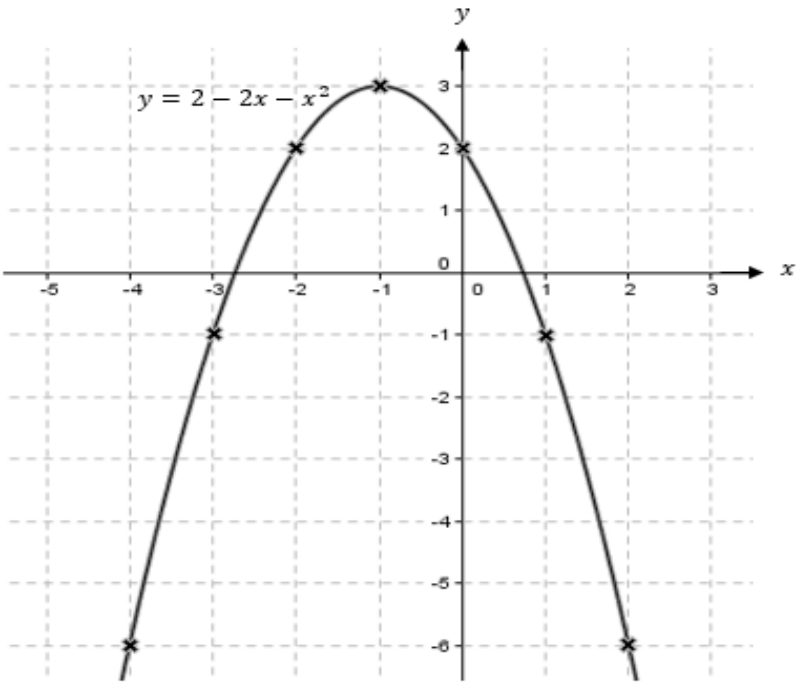


Fig 24.2

The quadratic equation or function can either have a maximum turning point or a minimum turning point

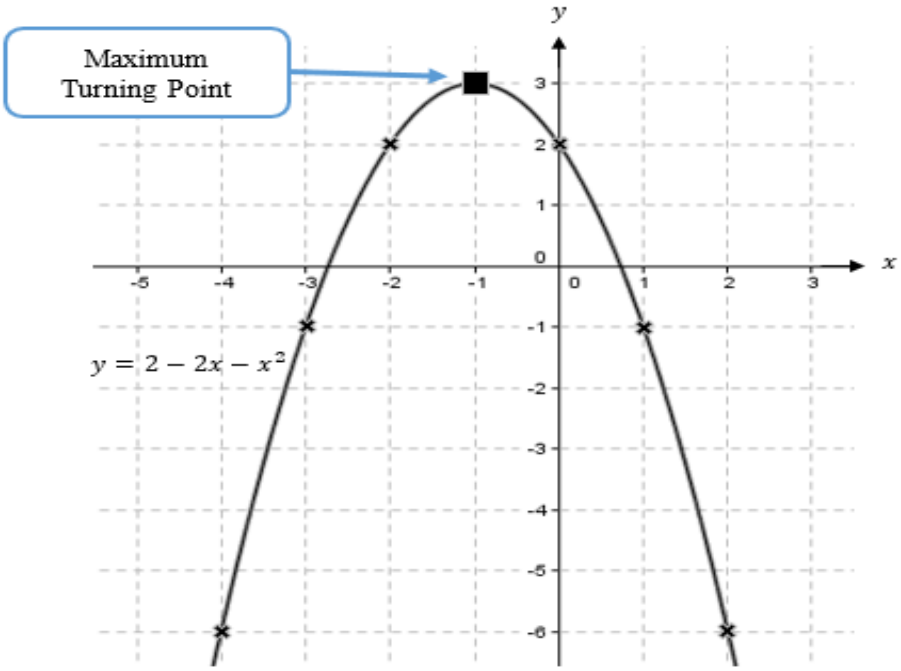


Fig 24.3

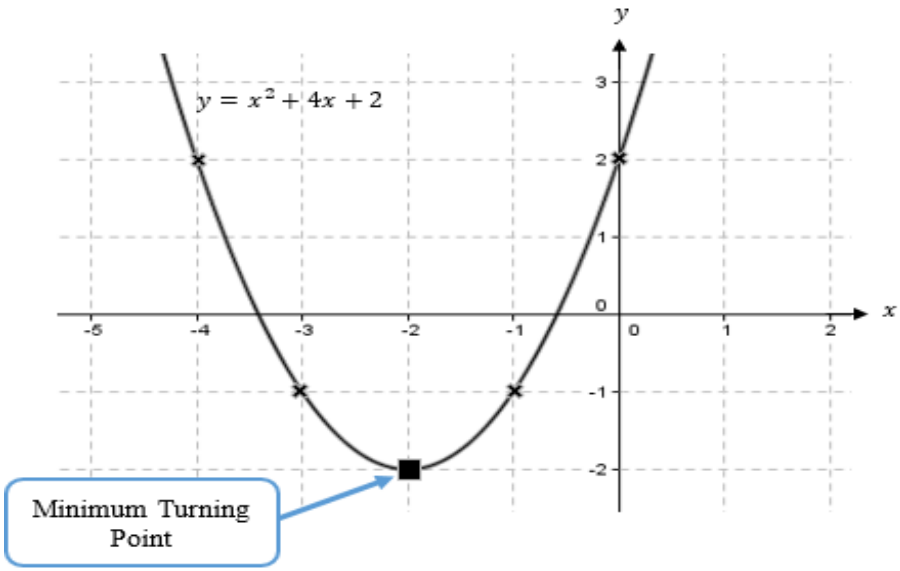


Fig 24.4

24.2.2 Plotting quadratic graphs/functions

Do you still remember how we plotted a straight line? Well we started with a table of values. The same applies to the plotting of a quadratic function.

Step 1 – Start by making out a table of values

Step 2 – Using the given scale or an appropriate scale we mark coordinates in the Cartesian plane

Step 3 – Join the points with a smooth curve (here you do not join the points using a ruler)

Let us consider the following example and find out how to plot a quadratic function

Worked Example [1]

Questions

a) Copy and complete the table of values for $y = x^2 - 4x + 4$

x	0	0,5	1	1,5	2	3	4
y	4			0,25			4

b) Using a scale of 2cm to represent 0,5 units on the horizontal axis and 2cm to represent 1 unit on the vertical axis, draw the graph of $y = x^2 - 4x + 4$

Solution

a) To complete the table of values, you have to substitute the value of x which correspond to the missing value of y

When $x = 0.5$; y is found by substituting x by 0.5 in the equation $y = x^2 - 4x + 4$

$y = (0,5)^2 - 4(0,5) + 4$

$y = (0,25) - 2 + 4$

$y = 2.25$

When $x = 1$; y is found by substituting x by 1 in the equation $y = x^2 - 4x + 4$

$y = (1)^2 - 4(1) + 4$

$y = 1 - 4 + 4$

$y = 1$

When $x = 2$; y is found by substituting x by 2 in the equation $y = x^2 - 4x + 4$

$y = (2)^2 - 4(2) + 4$

$y = 4 - 8 + 4$

$y = 0$

When $x = 3$; try this yourself. Find the value of y in the space below

If you did this correctly your answer should be 1

Fill in the table of values to have the completed table of values as follows

x	0	0.5	1	1.5	2	3	4
y	4	2.25	1	0.25	0	1	4

b) Using the given scale, mark the coordinates as shown below

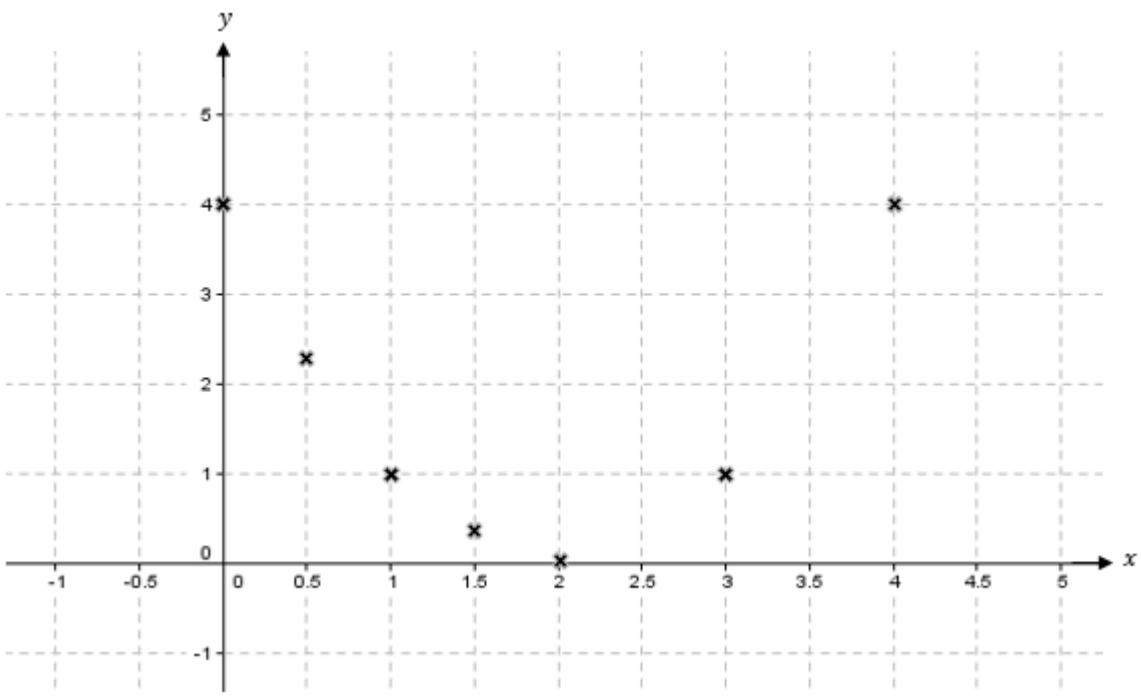


Fig 24.5

Having marked the coordinates, join the points with a smooth curve and label the curve (do not use ruler to join the points)

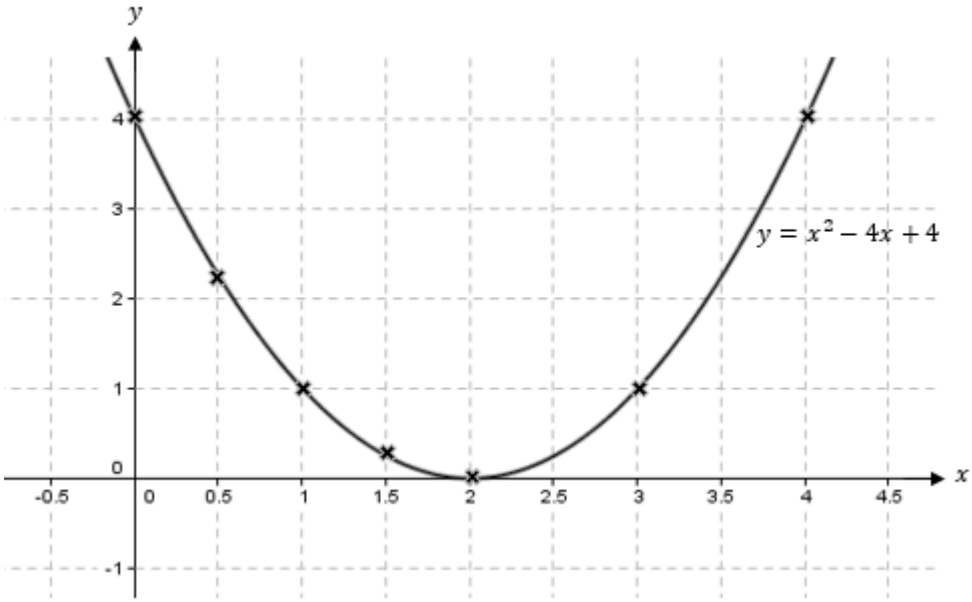


Fig 24.6

Let us consider another example in which you will be doing much of the working.

Worked Example [2]

Questions

a) Find the missing values in the table of values for $y = 3 - 2x - x^2$

x	–4	–3	–2	–1	0	1	2
y	–5	a	b	c	3	0	–5

b) Using a scale of 2cm to represent 1 unit on both axis, draw the graph of $y = 3 - 2x - x^2$

Solutions

a) When $x = -3$; a is found by substituting the value of -3 in the equation $y = 3 - 2x - x^2$

$a = 3 - 2(-3) - (-3)^2$

$a = 3 + 6 - 9$

$a = 9 - 9$

$a = 0$

Now use the space provided to find the value of b and c

If you have done this correctly, your table of values should be as follows,

x	–4	–3	–2	–1	0	1	2
y	–5	0	3	4	3	0	–5

b) The points have been plotted for you in the following Cartesian plane, join with a smooth curve to make the curve and label the curve

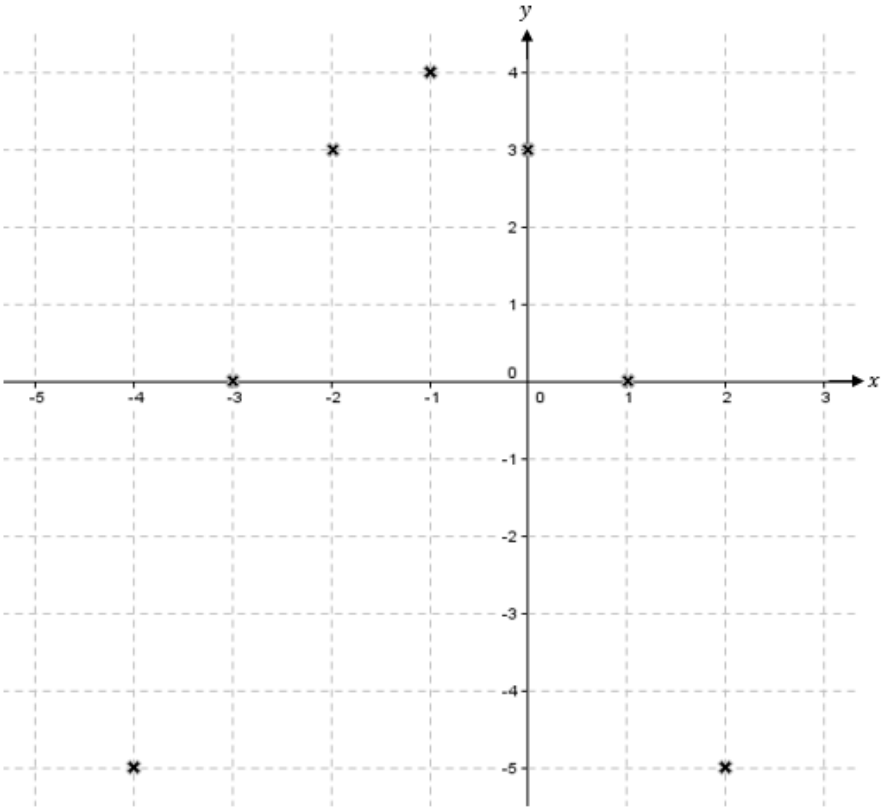


Fig 24.7

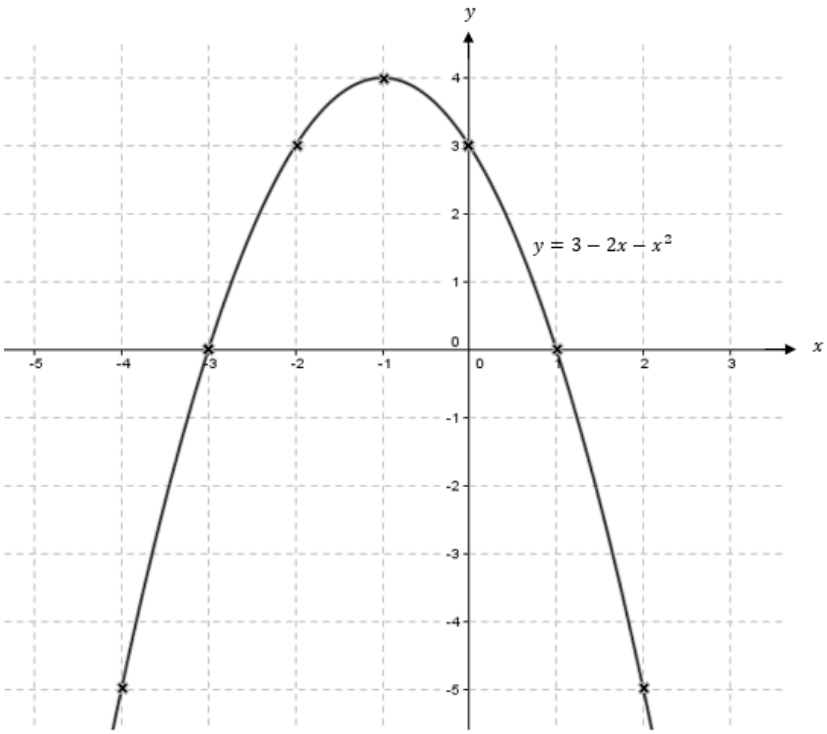


Fig 24.8

Activity (*24.1*) Plotting a quadratic graph

Questions

1) a) Copy and complete the table of values for $y = x^2$

x	–4	–3	–2	–1	0	1	2	3	4
y	16	9	m	n	0	1	p	q	16

c) Using a scale of 2cm to represent 1 unit on the x-axis and 2cm to represent 4 units on the y axis, plot the graph of $y = x^2$

2) The following is a table of values for $y = x^2 + 4x - 22$

x	–8	–7	–6	–4	–2	0	2	3	4
y	10	–1	–10	–22	–26	–22	–10	–1	10

Using a scale of 2 cm to represent 2 units on the x axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = x^2 + 4x - 22$, for $-8 \leq x \leq 4$ and $-30 \leq y \leq 15$

Tip – in the x-axis start from -8 to 4 ($-8 \leq x \leq 4$) and in the y-axis start from 15 to -30 ($-30 \leq y \leq 15$)

3) a) Copy and complete the table of values for $y = 2x^2 + 5x - 4$

x	–4	–3	–2	–1	0	1
y	8	a	b	c	–4	3

b) using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 2 units on the y –axis , draw the graph of $y = 2x^2 + 5x - 4$

4. Draw the graph of $y = 6x - 2x^2$ from $x = -1$ to $x = 5$, use a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to 1 unit on the y-axis

Answers

1) (a) m= 4 n= 1 p= 4 q= 9

(b)

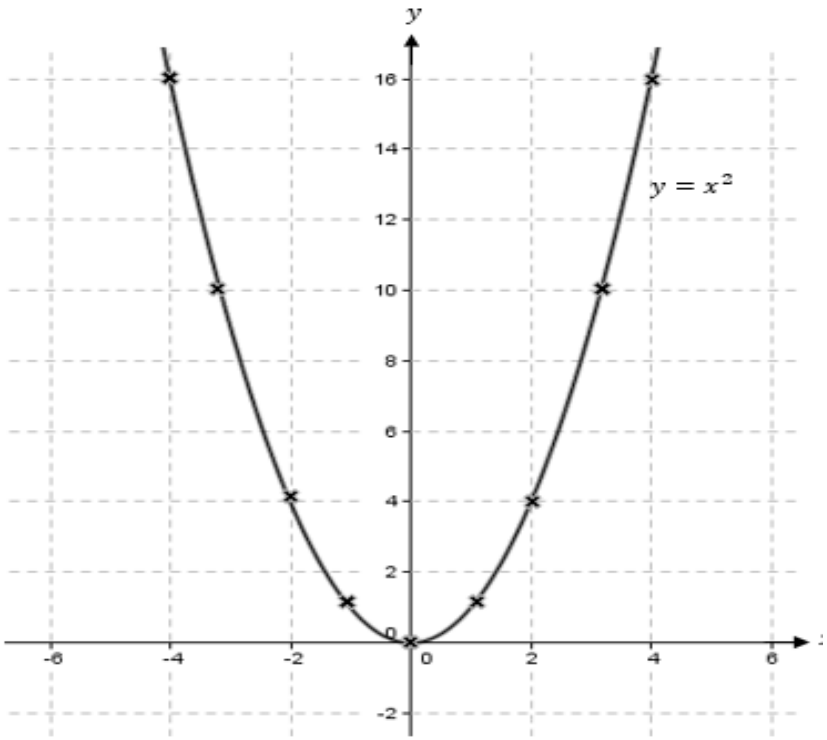


Fig 24.9

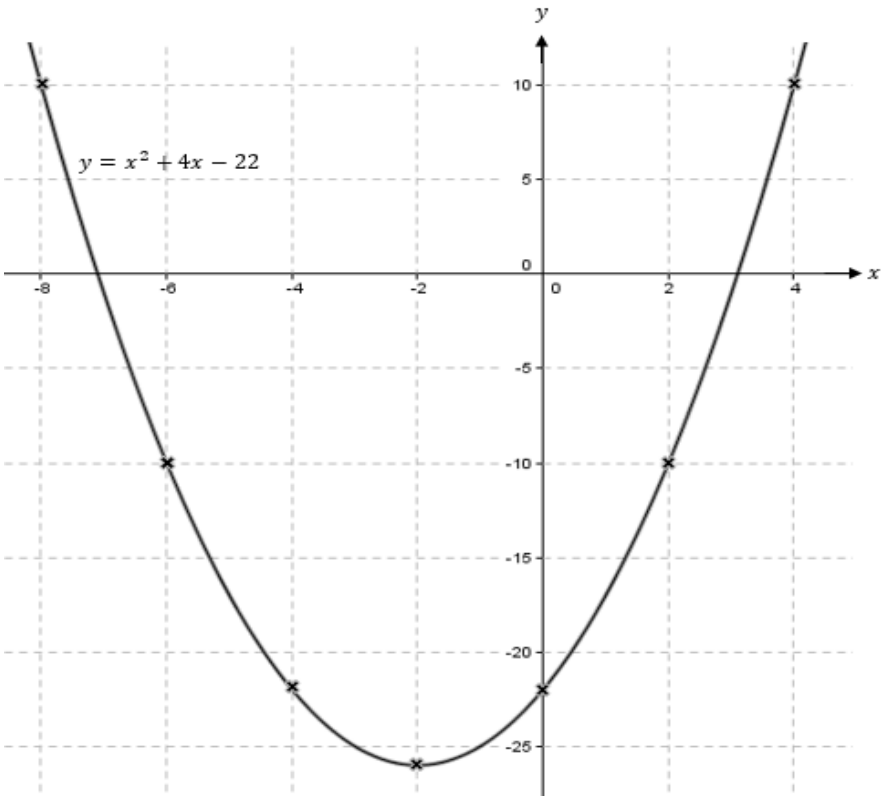


Fig 24.10

2) (a) a = –1 b = –6 c = –7

(b)

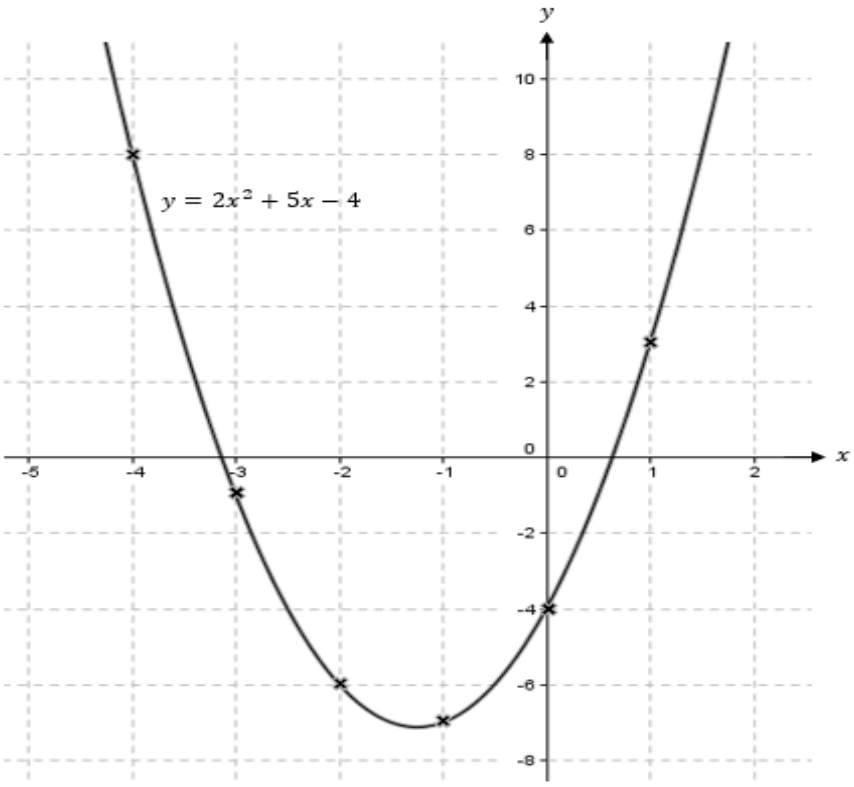
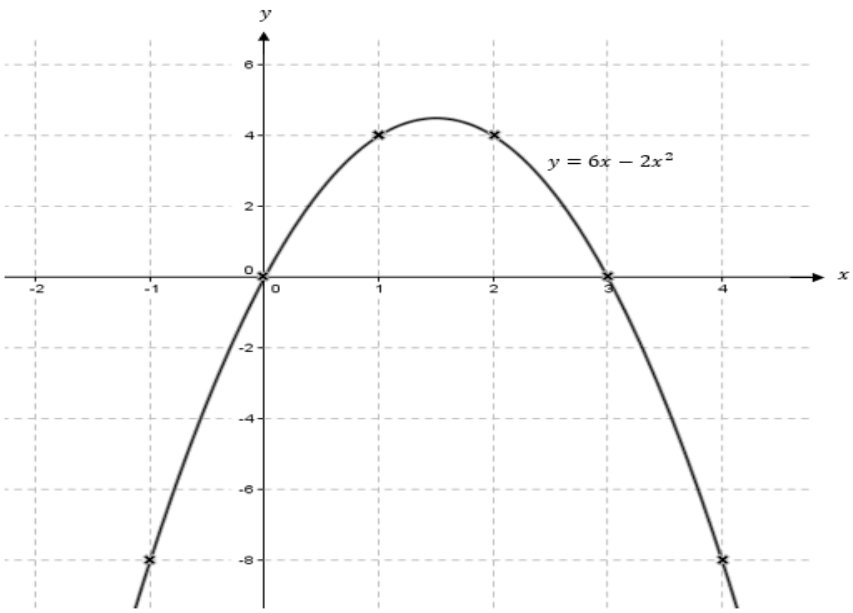


Fig 24.11

3)



24.2.3 Area under the curve (under the quadratic graph)

The area under a curve is the area contained between the curve and the axis and or other given straight lines.

Look at the graph below; the shaded area represents area under the curve

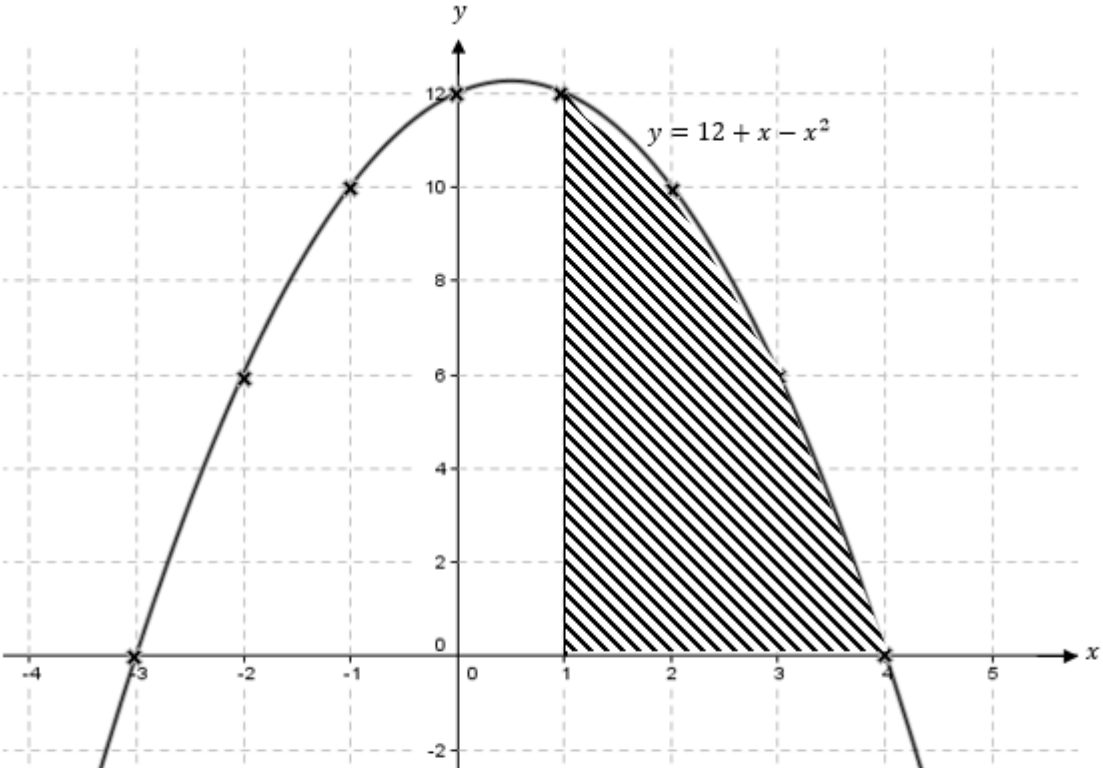


Fig 24.9

There are basically two methods of calculating the area under the curve which are;

- Counting the squares
- Splitting the area into trapeziums and calculate the summation of the trapeziums

a) Counting the squares

Here some points to remember when using the method of counting squares

- Decide the area of each small square on the graph, thus take note of the scale.
- Incomplete squares are counted if their area is greater than half a square and ignore if their area is half a square

Let us consider the following example on how to calculate area using the method of counting squares

Worked Example [3]

Questions

a) Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 2 units on the y –axis, draw the graph of $y = 12 + x - x^2$

b) Find the area enclosed by the curve, the x –axis , $x = 1$ and $x = 3$

Solutions

a)

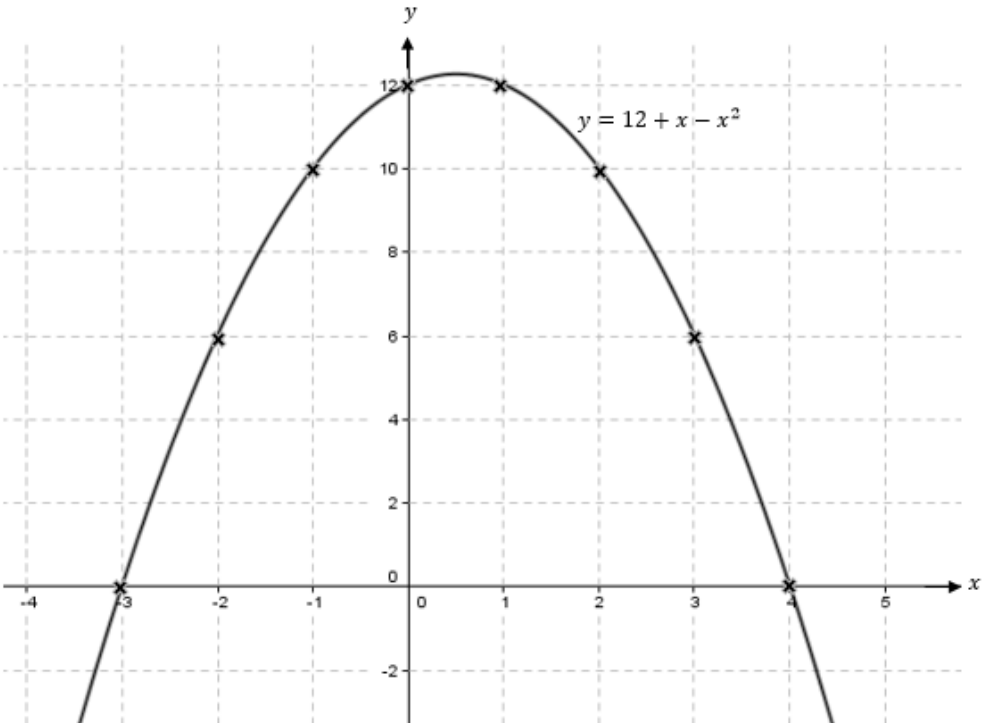


Fig 24.10

b)

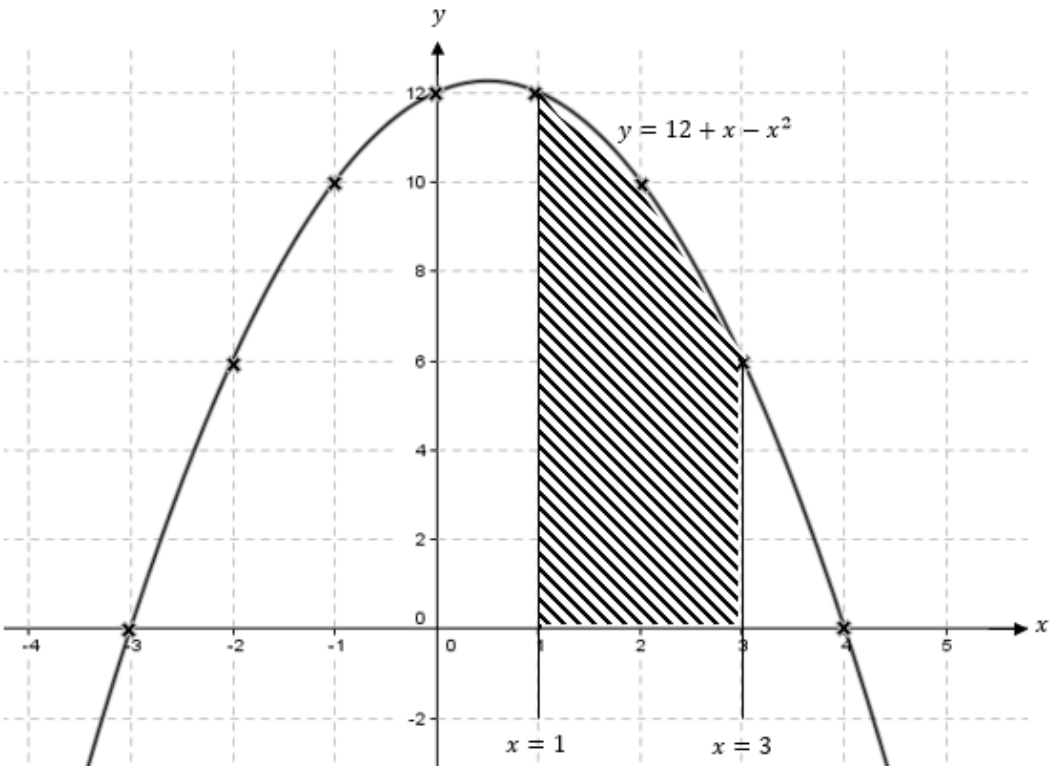


Fig 24.11

Step 1: Indicate the space whose area is to be calculated by drawing the lines $x = 1$ and $x = 3$,

Step 2: Count the number of squares

Step 3: Multiply the number of the squares found by the area of the small square which is

$(\frac{2}{10} \times \frac{1}{10} \times$ number of squares)

The area of a single square is $\frac{1}{10} \times \frac{2}{10}$ (you get this from the given scale)

Worked Example [4]

Questions

The following is the table of values for $y = x^2 - 4x + 4$

x	0	0,5	1	1,5	2	3	4
y	4	2,25	1	0,25	0	1	4

a) Using a scale of 2 cm to represent 0.5 unit on the x −axis and 2 cm to represent 1 unit on the y- axis , draw the graph of $y = x^2 - 4x + 4$

b) Find the area bounded by the curve, the x −axis, $x = 0{,}5$ and $x = 1$

Solutions

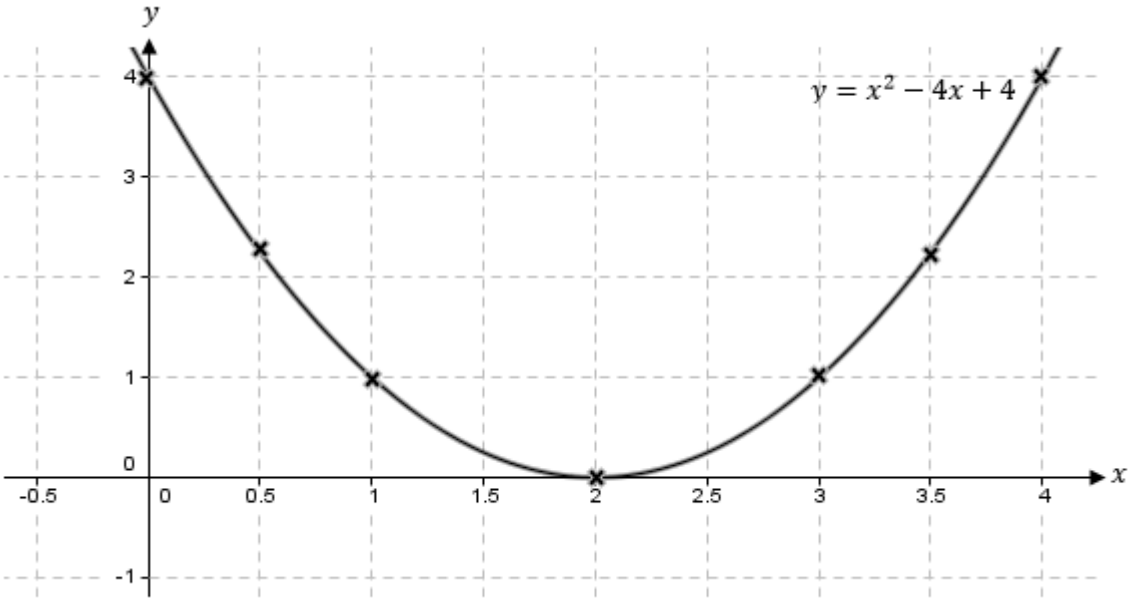


Fig 24.12

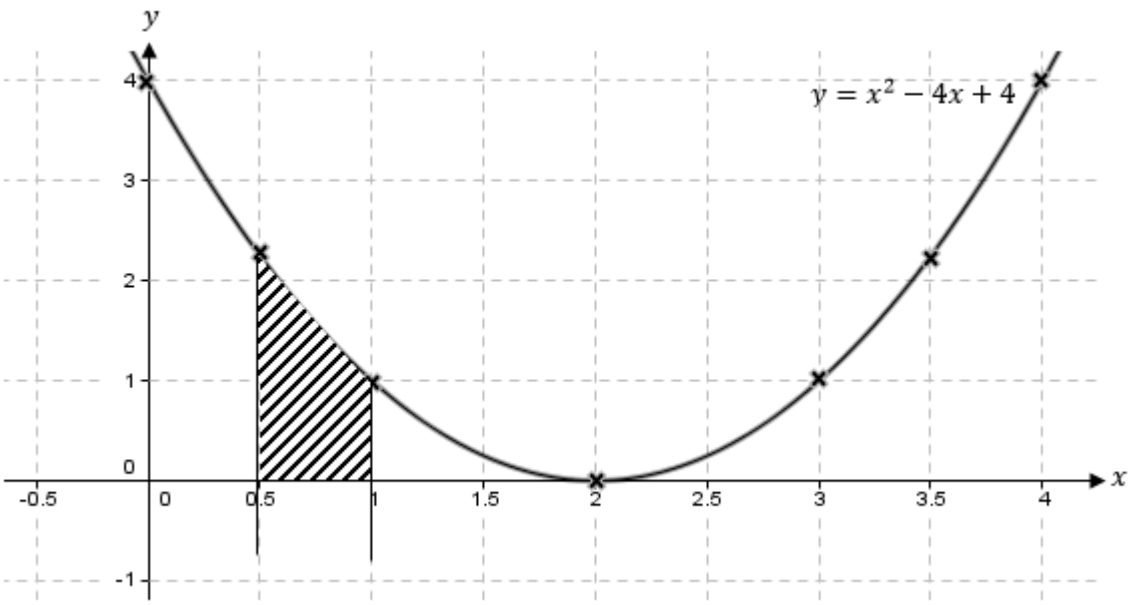


Fig 24.13

Step 1 – Indicate by drawing the lines $x = 0.5$ and $x = 1$, the space whose area is to be calculated

Step 2 – Count the number of squares

Step 3 – Multiply the number of the squares found by the area of the small square which is

$$\frac{5}{100} \times \frac{1}{10} \times \text{number of squares}$$

$$0.05 \times 0.1 \times 150 = 0.75 \text{ units}^2$$

Let us consider the following example to find the area using the method of trapeziums

Using the trapeziums

Now let us find the area of the same functions above using the method of trapeziums. Sometimes we can split the area under the curve in different shapes such as a triangle, rectangle and squares together with the trapezium.

We simply add the area of the different shapes obtained using their respective formulas.

Worked Example [5]

Questions

a) Using a scale of 2 cm to represent 1 unit on the x-axis and 2cm to represent 2 units on the y- axis, draw the graph of $y = 12 + x - x^2$ for $-3 \leq x \leq 4$

b) Find the area enclosed by the curve, the x-axis, $x = 1$ and $x = 3$

Solutions

a)

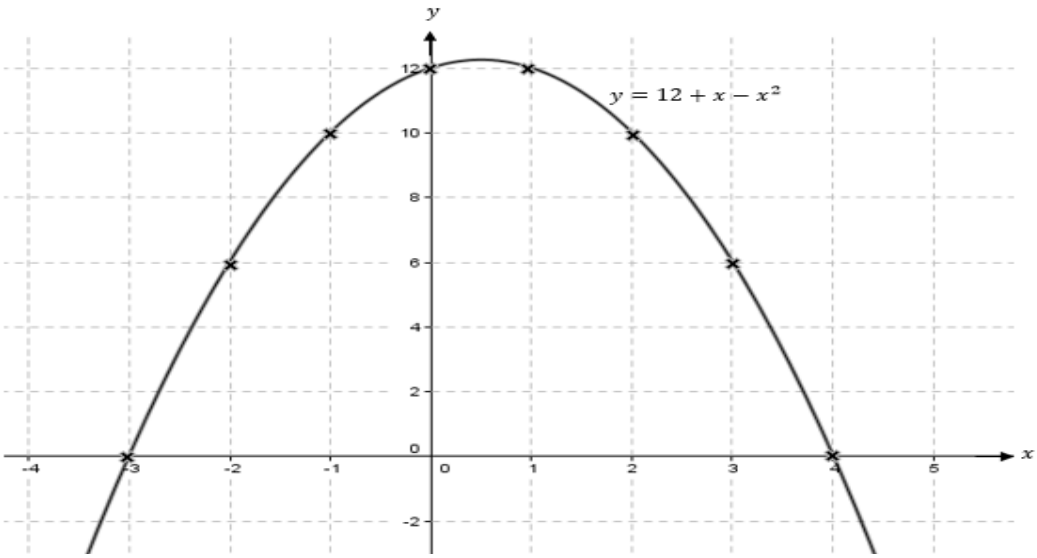


Fig 24.14

b) show on the graph the trapezium from $x = 1$ to $x = 2$ and another trapezium from $x = 2$ to $x = 3$

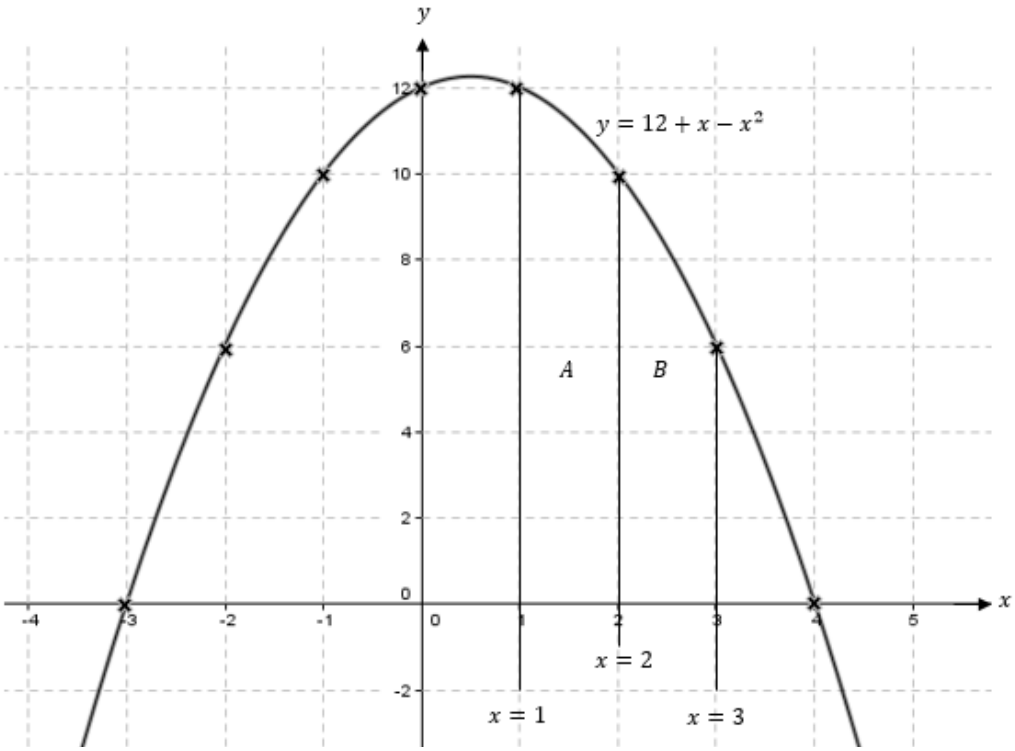


Fig 24.15

Calculate the area of A and of B and sum the area.

Do this yourself

The following is the table of values for $y = x^2 - 4x + 4$

x	0	0,5	1	1,5	2	3	4
y	4	2,25	1	0,25	0	1	4

a) Using a scale 2 cm to represent 0,5 units on the x −axis and a scale of 2cm to represent 1 unit on the y −axis, draw the graph of $y = x^2 - 4x + 4$

Fig 24.16

b) Calculate the area enclosed by the curve, x-axis, $x = 1$ and $x = 4$ (using the method of trapeziums)

Use the space below to calculate the area

24.2.4 Gradient of a Quadratic function

What is a tangent? Yes! It is the line that touches the curve at just one particular point. See examples below

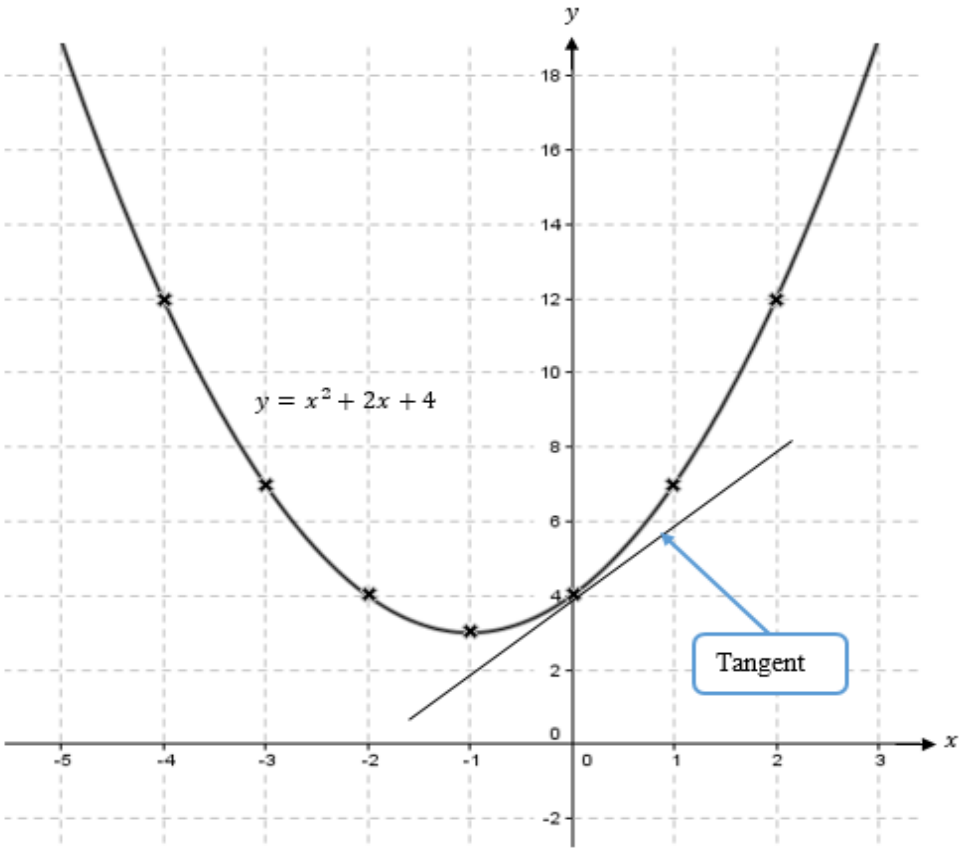


Fig 24.17

Gradient can be negative or positive

Positive gradient line

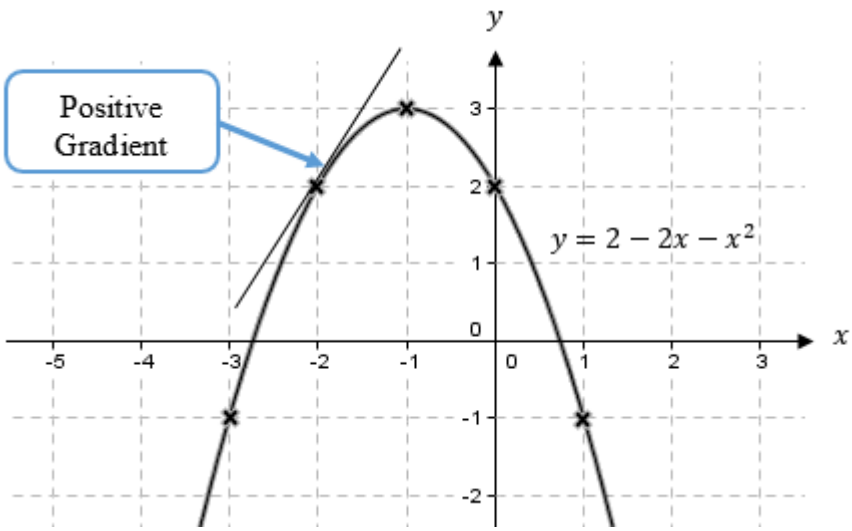


Fig 24.18

Negative gradient line

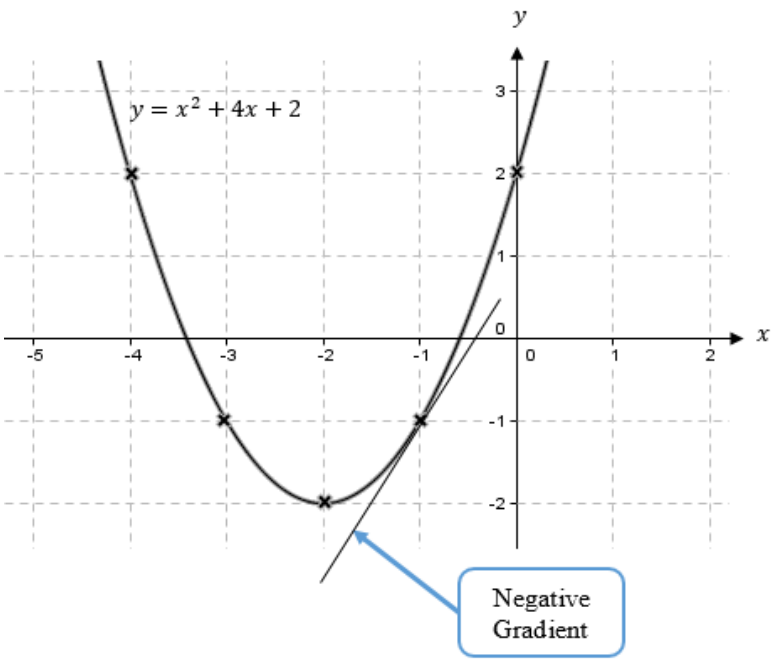


Fig 24.19

The gradient at any point on a curve is equal to the gradient of the tangent to the curve at that point.

Let us consider the following example to find the gradient at a point. Hint; Calculation of gradient is done after plotting the quadratic graph

Worked Example [6]

Questions

Using the graph of $y = x^2 - 4x + 4$ below, estimate using your graph the gradient of the curve at $x = 0.5$

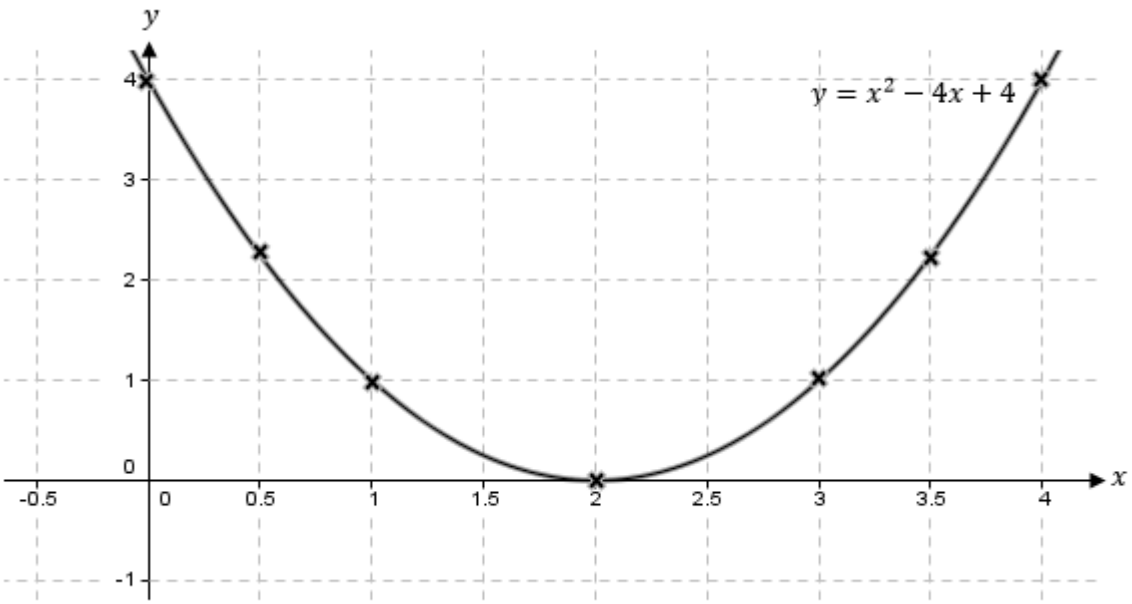


Fig 24.20

Solutions

On the graph above, draw a tangent at $x = 0.5$ as shown below

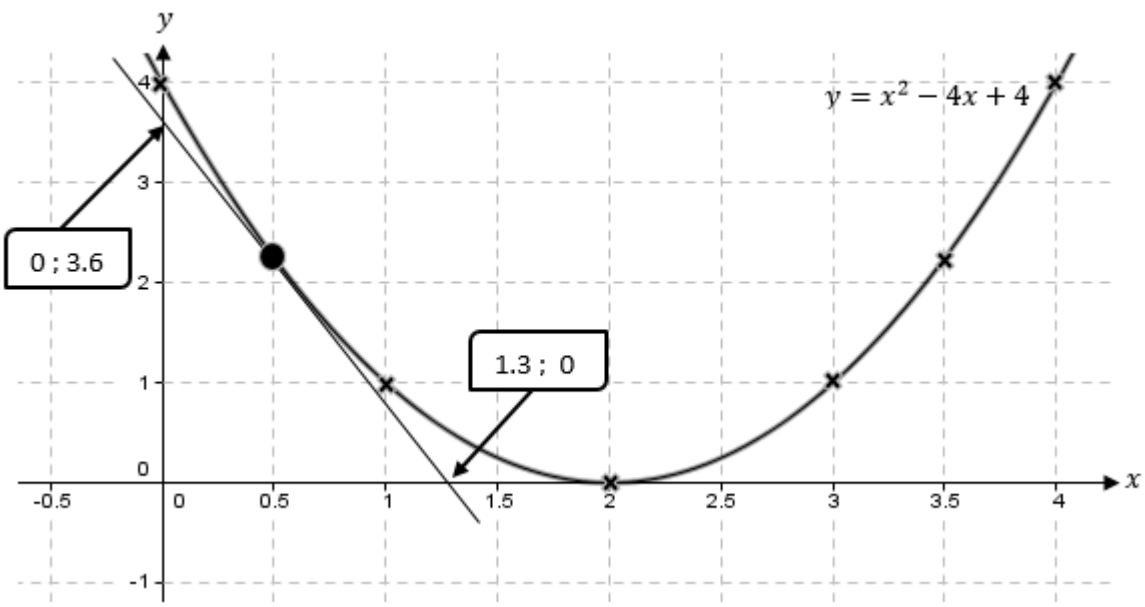


Fig 24.21

Having drawn the tangent use any method you wish to use to find the gradient of the tangent line.

Use of two coordinates from the tangent	Use of a right angled triangle
The coordinates are (0; 3.6) and (1.3 ; 0) Grad = $\frac{3,6-0}{0-1,3} = \frac{3,6}{-1,3} = -2.77$	Gradient = $\frac{3,6}{1,3} = 2.77$ but remember that these line have a negative gradient Therefore the gradient –2.77

Worked Example [7]

Questions

Using the graph of $y = 3 - 2x - x^2$ for $-4 \leq x \leq 2$ and $-5 \leq y \leq 5$ and using a scale of 2cm to represent 1 unit on both axis, find the gradient of the curve at $x = -2$

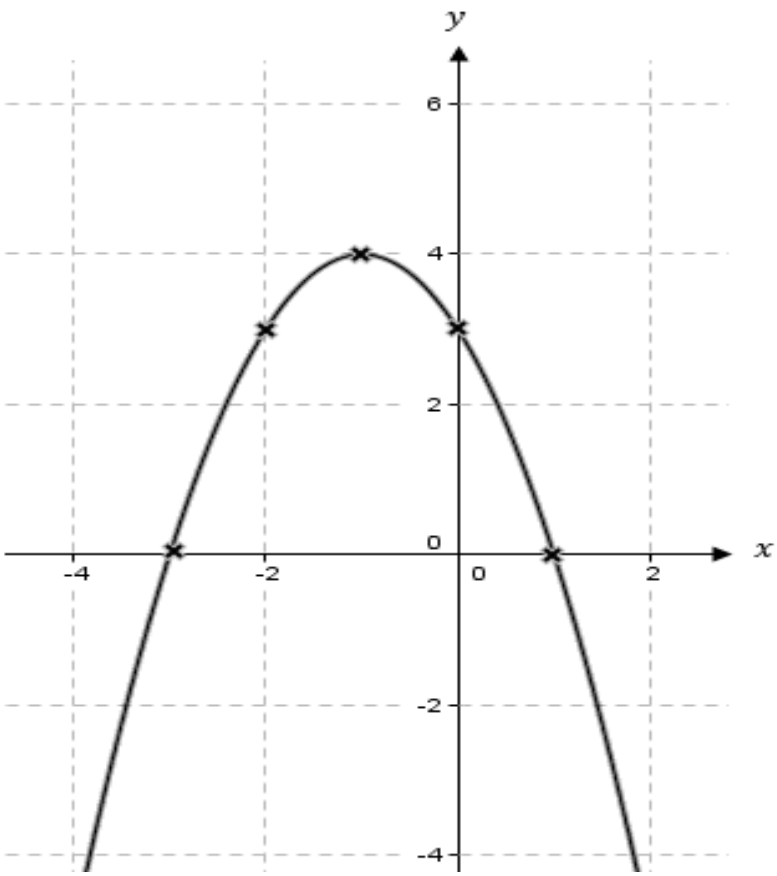


Fig 24.22

Solutions

On the graph above, draw the tangent at $x = -2$ and draw a right angled triangle as shown below and calculate the gradient

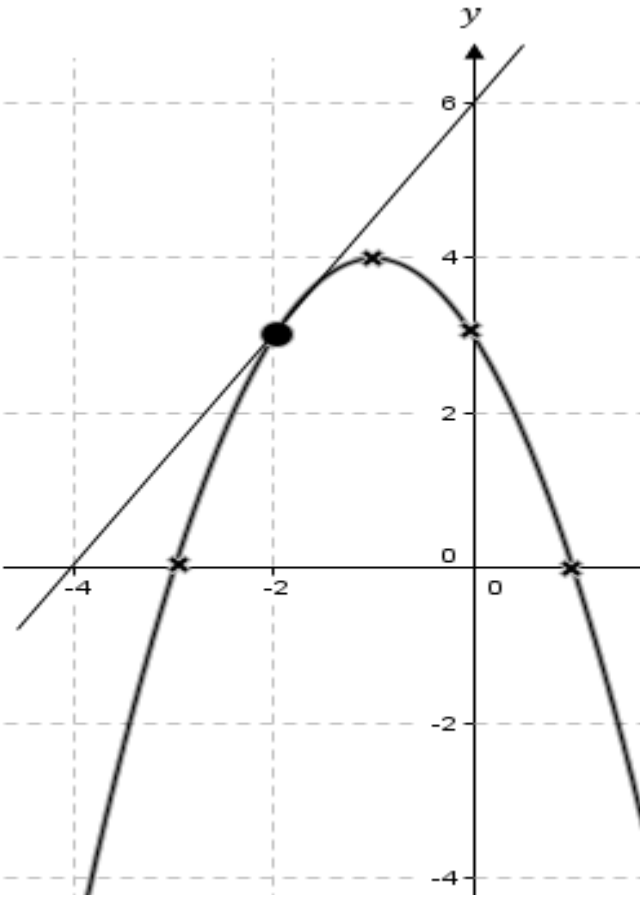


Fig 24.23

Gradient $= \frac{6}{4} = 1.5$

Do this yourself

Using the graph below of $y = 12 + x - x^2$, find the gradient of the curve at $x = 2$

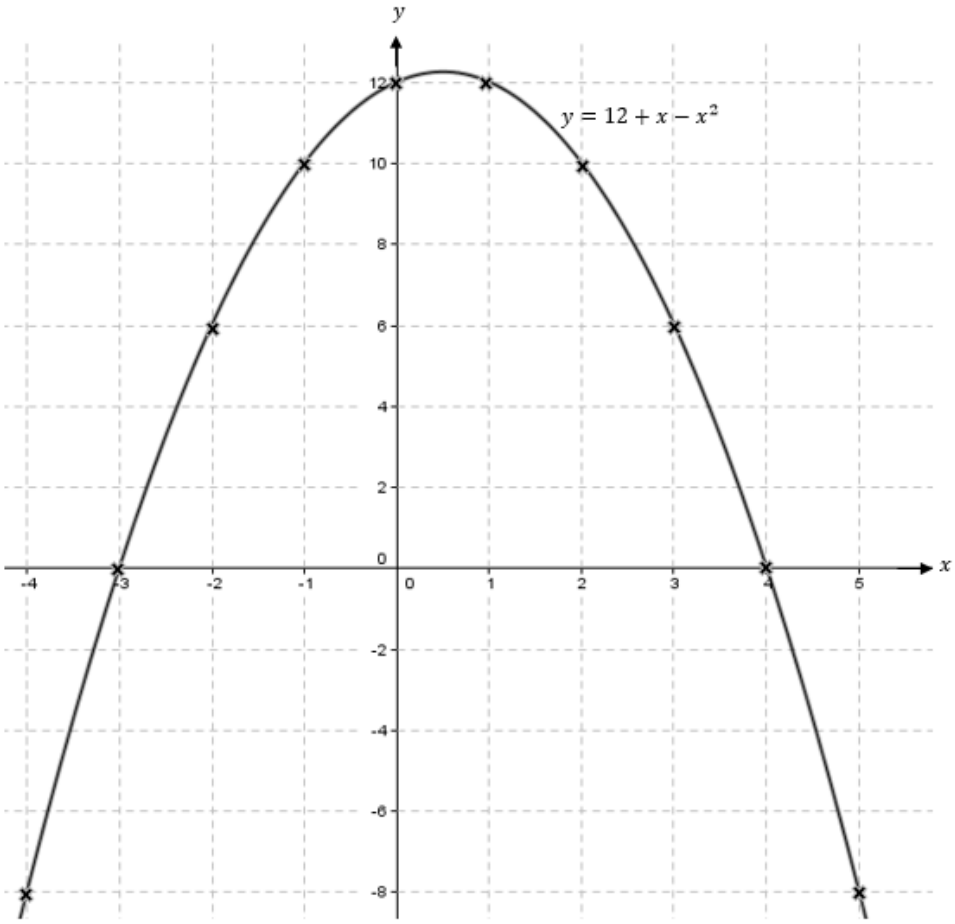


Fig 24.24

24.2.5 Roots of a quadratic function

The roots of a quadratic function or graph are the values of x for which the graph intersects another curve or straight line. Let us consider the following examples on finding the roots.

Worked Example [8]

Questions

Given the graph of $y = 3 - 2x - x^2$ below, find the roots of the equation

(a) $0 = 3 - 2x - x^2$

(b) $2 = 3 - 2x - x^2$

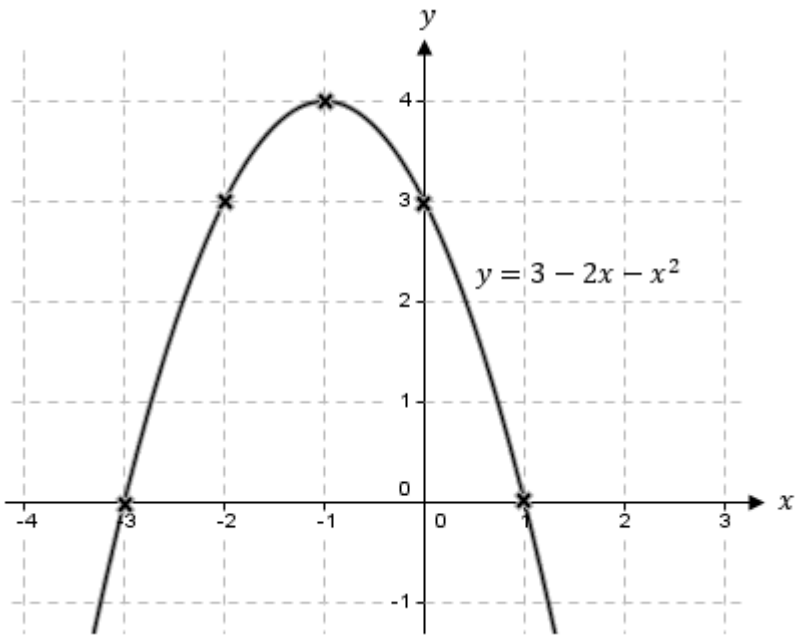


Fig 24.25

Solution

Subtract the equation ($0 = 3 - 2x - x^2$) from the original equation ($y = 3 - 2x - x^2$)

$$\begin{array}{rl} & y = 3 - 2x - x^2 \\ - & 0 = 3 - 2x - x^2 \\ \hline & y - 0 = 3 - 3 + -2x - (-2x) - (-x^2) - (-x^2) \\ & y = 0 \\ \hline \end{array}$$

Now you draw the graph of $y = 0$ on the graph above as shown below

Where the graph of $y = 0$ intersect the original graph thus where the roots are found.

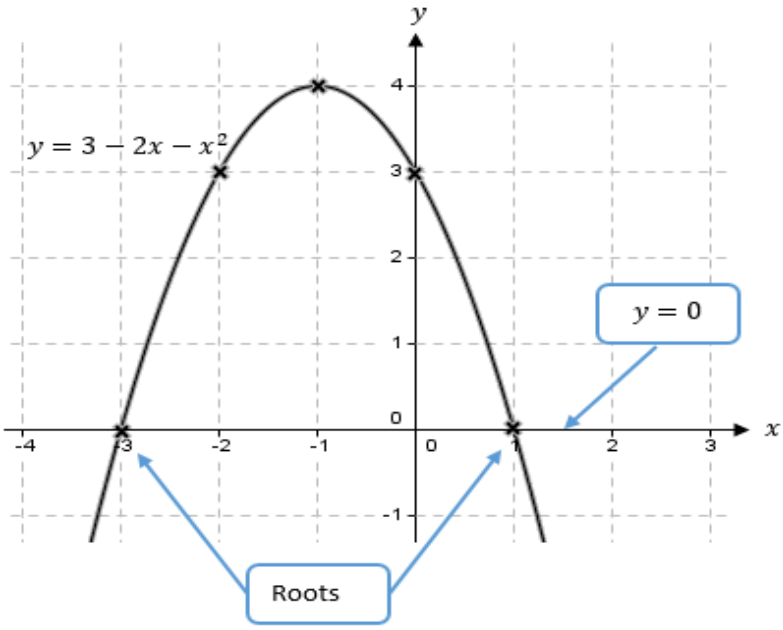


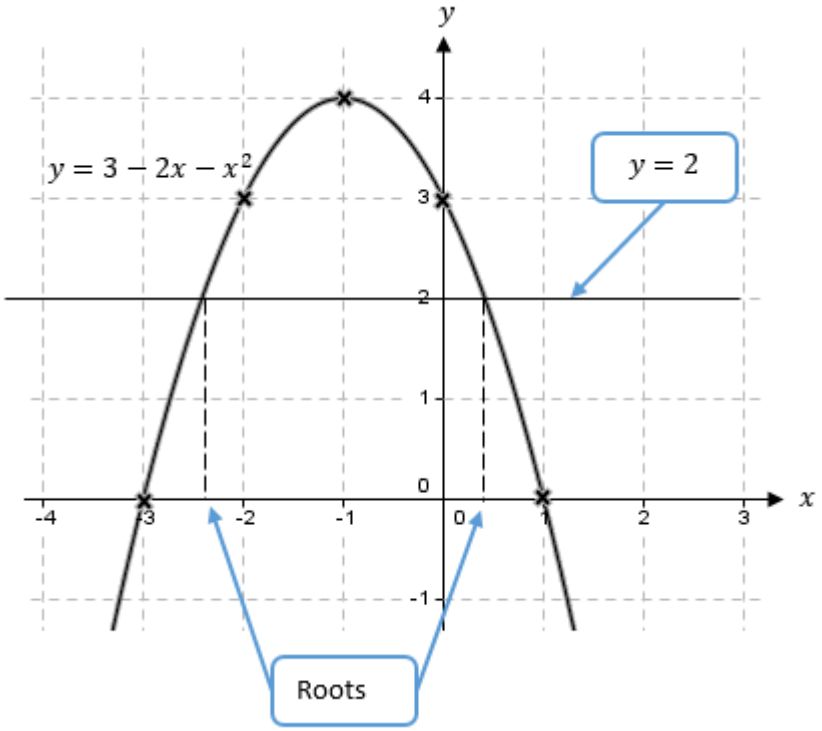
Fig 24.26

The roots are $x = -3$ or $x = 1$

(c) Subtract the equation ($2 = 3 - 2x - x^2$) from the original equation ($y = 3 - 2x - x^2$)

$$\begin{array}{rl} & y = 3 - 2x - x^2 \\ - & 2 = 3 - 2x - x^2 \\ \hline & y - 2 = 3 - 3 + -2x - (-2x) - (-x^2) - (-x^2) \\ & y - 2 = 0 \quad \text{Make y the subject of the formulae} \\ \hline & y = 2 \end{array}$$

Now you draw the line of $y = 2$ as shown below. Where the line intersects with the original graph thus where we find the roots of the equation.



Now let us consider the intersection of the graph with slant or oblique lines.

Worked Example [9]

Questions

The following is the graph of $y = x^2 + 5x - 7$

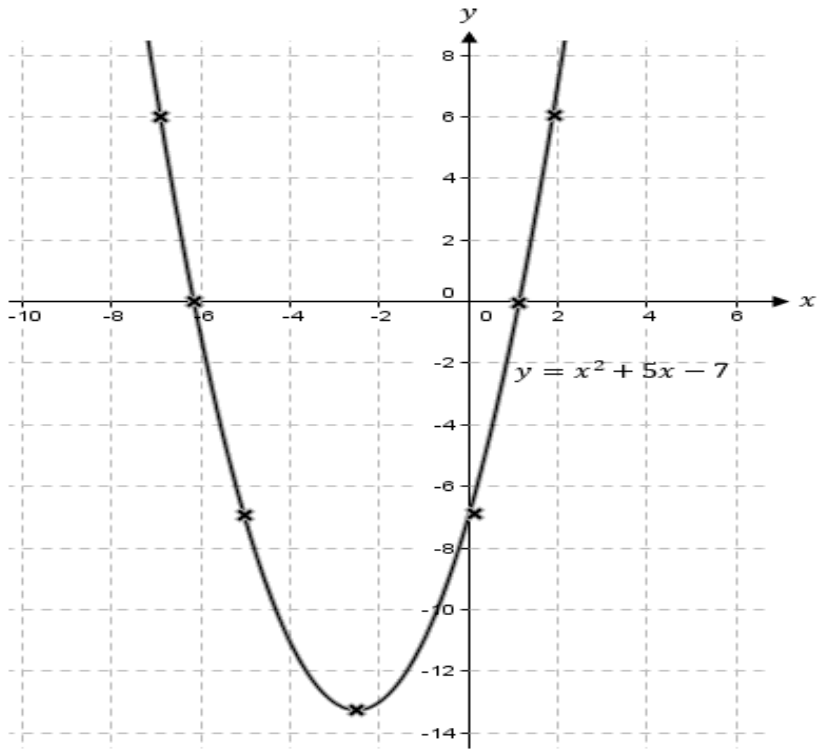


Fig 24.28

a) Draw the graph of $y = 2x + 2$

b) Find the roots of the equation $x^2 + 5x - 7 = 2x + 2$

Solution

a) To plot a straight line, we construct a table of values first

x	–2	–1	0	2
y	–2			

When $x = -1$;

$$y = 2(-1) + 2$$
$$y = -2 + 2$$
$$y = 0$$

When $x = 0$;

$$y = 2(0) + 2$$
$$y = 0 + 2$$
$$y = 2$$

When $x = 2$ now try to work this out on the space below

Your table of values should be similar to this one below

x	–2	–1	0	2
y	–2	0	2	6

Now you plot the straight line. Plot your straight line on the graph above as shown in the diagram below

Alternatively you can plot a simple table of values for a straight line. You just use $x = 0$ and $y = 0$

x	0	
y		0

When $x = 0$;

$y = 2(0) + 2$

$y = 2$

When $y = 0$,

$0 = 2(x) + 2$

$-2 = 2x$ (now we divide by 2)

$\frac{-2}{2} = \frac{2x}{2}$

$-1 = x$

x	0	–1
y	2	0

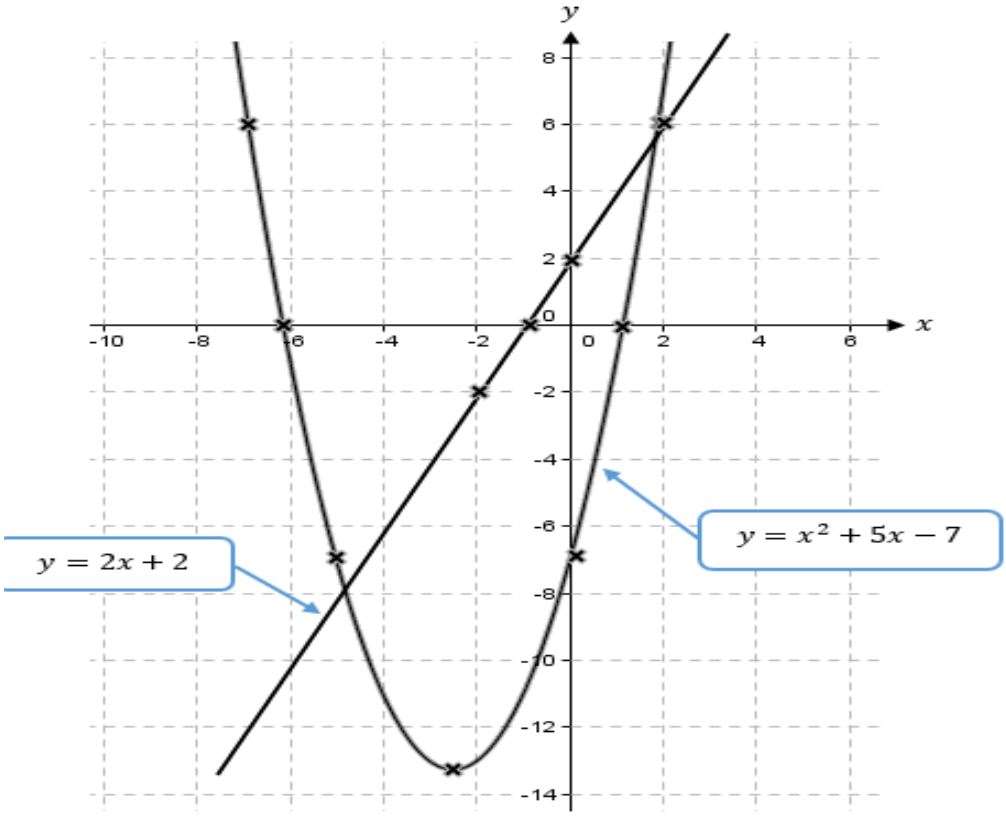


Fig 24.29

Now where the two graphs meet thus where we find the roots of the equation.

b) Finding the roots of the equation $x^2 + 5x - 7 = 2x + 2$

Do your working here in the space below,

Plot the straight line on the graph above, it should look like the one below

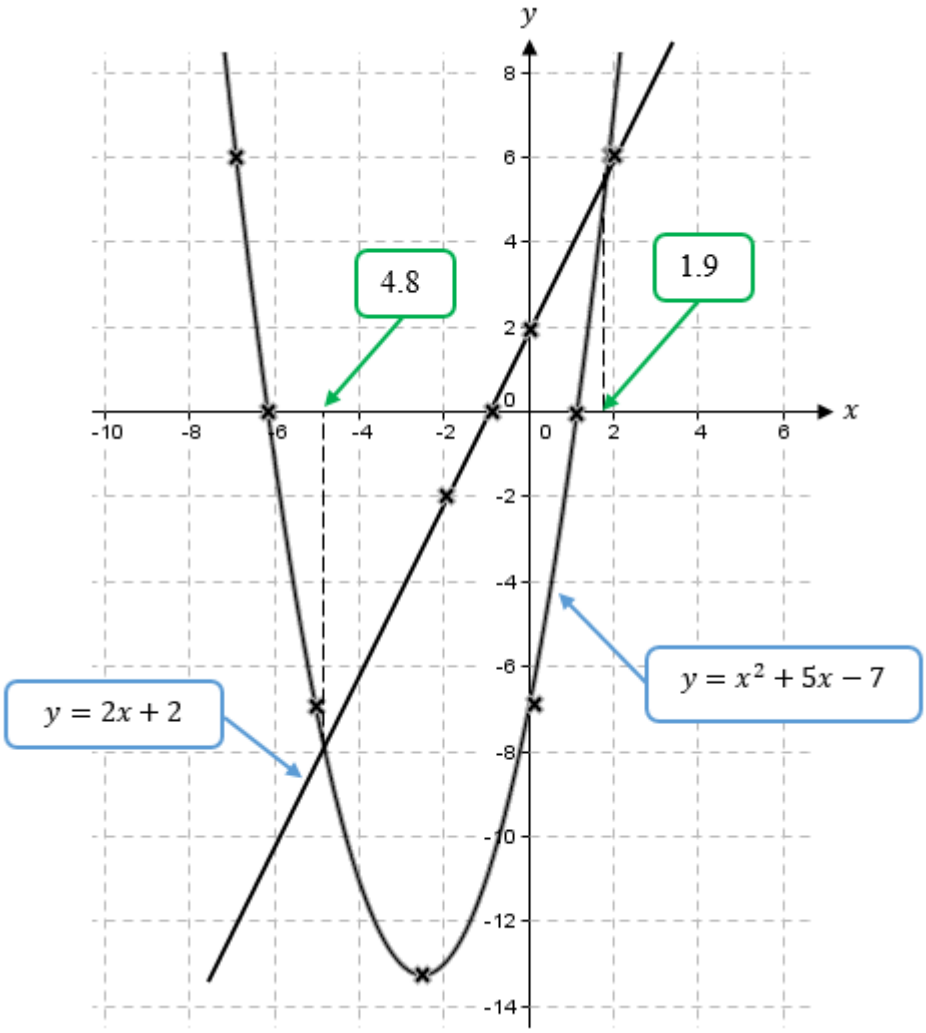


Fig 24.30

Now where the two graphs meet thus where we find the roots of the equation.

Worked Example [10]

Questions

The following is the graph of $y = 3 - 2x - x^2$

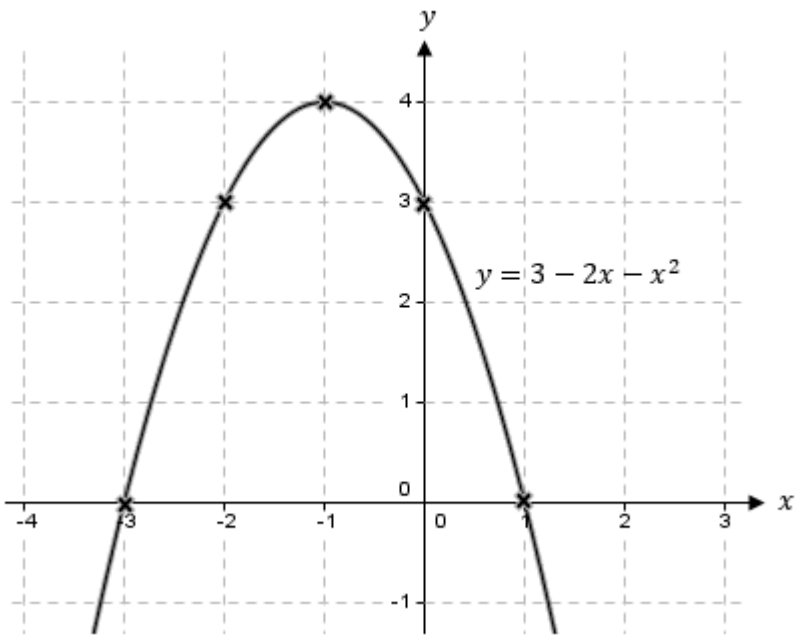


Fig 24.31

a) Find the roots of the equation $3 - 2x - x^2 = 0$

b) Solve the equation $3 - 2x - x^2 = -x + 2$

Solutions

a) Now we simply subtract the equation in question from the original equation.

$$\begin{array}{rl} & y = 3 - 2x - x^2 \\ - & 0 = 3 - 2x - x^2 \\ \hline & y - 0 = 3 - 3 + -2x - (-2x) - (-x^2) - (-x^2) \\ & y = 0 \\ \hline \end{array}$$

Draw the straight line on the graph above, it should look like the graph below

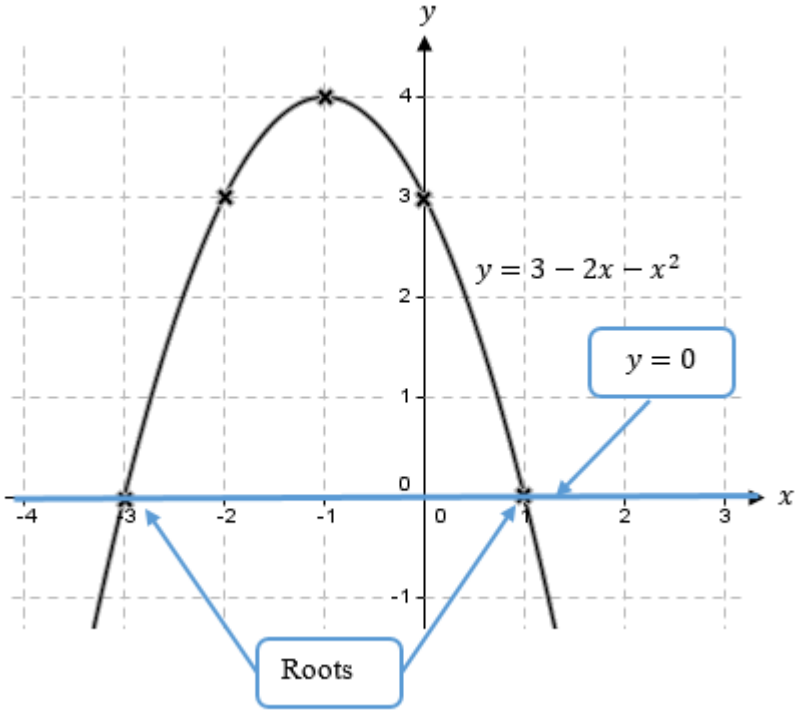


Fig 24.32

The roots are $x = -3$ or $x = 1$

Tip; At the point of intersection thus where we find our roots

b) We do the same procedure we used on a) above

$$\begin{array}{rl} & 3 - 2x - x^2 = y \\ - & 3 - 2x - x^2 = -x + 2 \\ \hline & 0 = y - (-x + 2) \\ & -x + 2 = y \\ \hline & y = -x + 2 \end{array}$$

Now you draw a table of values for the line $y = -x + 2$

x	0	2
y	2	0

Drawing the graph of $y = -x + 2$ on the graph above it should look like the one below.

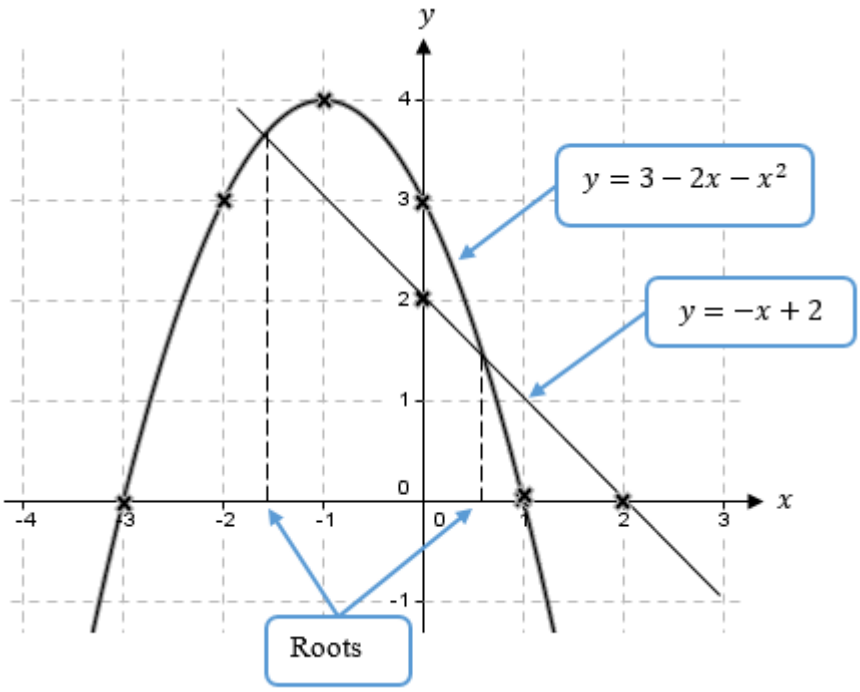


Fig 24.33

At the points of intersection thus where we find the roots (roots are taken from the x axis)

24.2.6 Maximum and Minimum

What do you understand by the term maximum? Yes it means greatest or highest. What do you think minimum mean? Yes! It means least or smallest. The turning point of a curve is either maximum or minimum.

At maximum turning point

The graph will look like the one below if the turning point is a maximum.

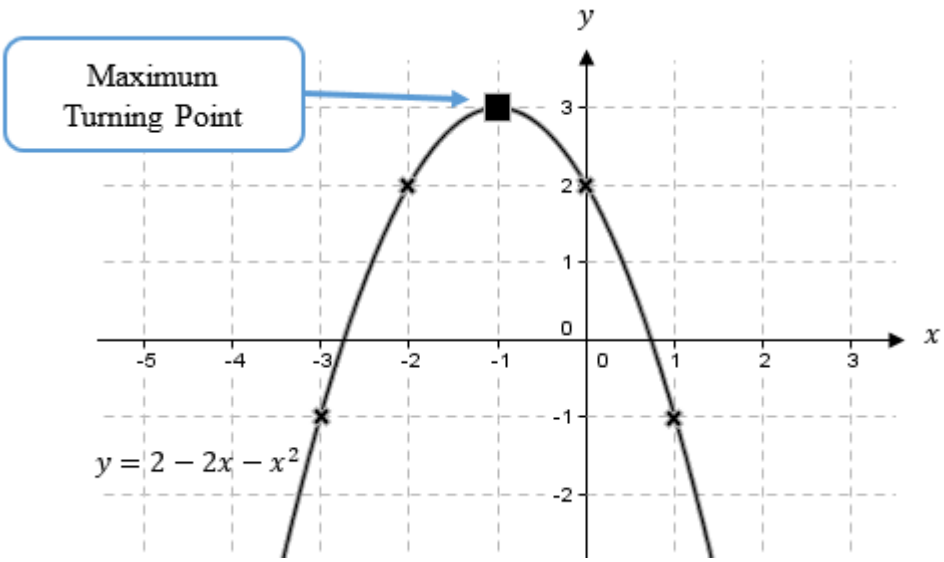


Fig 24.34

The coefficient of x^2 must be a negative for the graph to have a maximum turning point. On the graph, the coordinates of the turning point are (-1; 3)

At minimum turning point

The graph will look like the one below if the turning point is a minimum

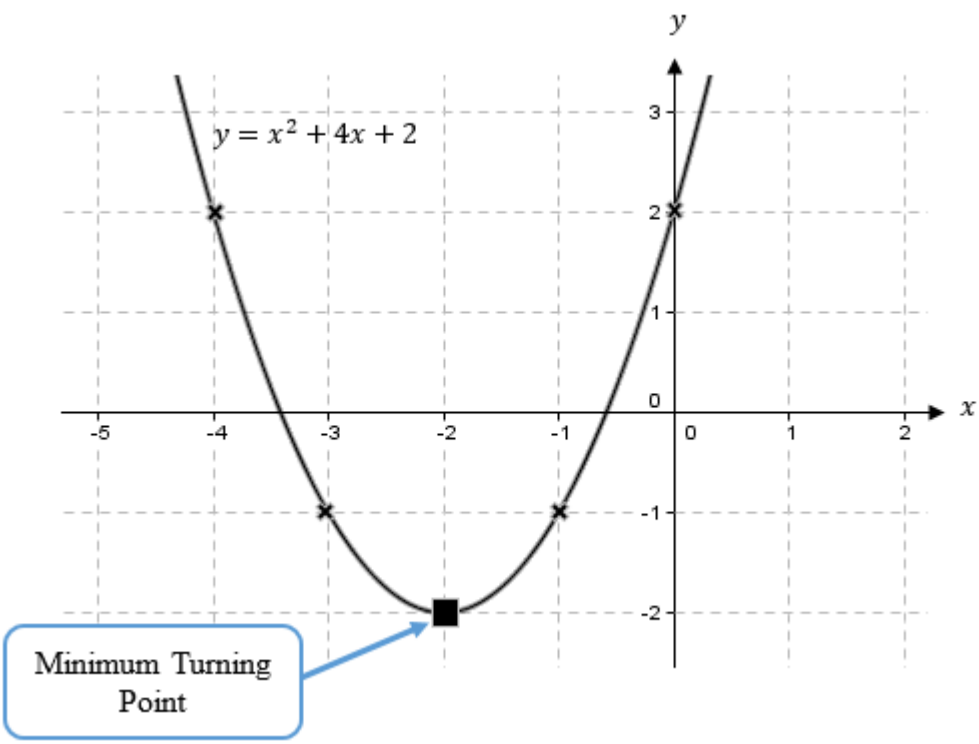


Fig 24.35

The coefficient of x^2 is always a positive for the function to have a minimum turning point

On the graph, the coordinates of the turning points are (–2; -2)

24.2.7 Line of symmetry

What is a line of symmetry? This refers to a line which cuts the graph into two equal parts

The line of symmetry is sometimes called the mirror line. The line of symmetry or the mirror line can be written as an equation known as the equation of the line of symmetry

Let us consider the following examples related to lines of symmetry.

Worked Example [11]

Questions

Draw using a dotted line the line of symmetry for the following graphs and on the graph, write the equation of the line of symmetry

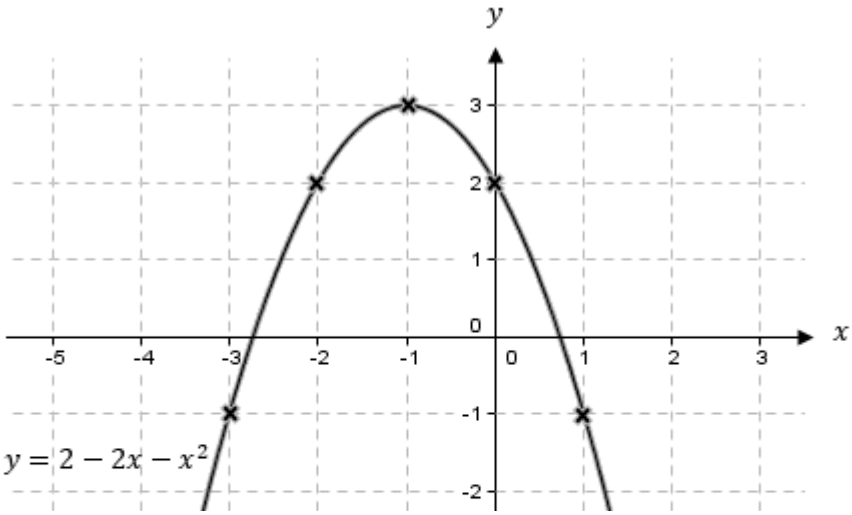


Fig 24.36

Solution

Draw the lines of symmetry on the above diagrams, the lines should look like the ones on the diagrams below.

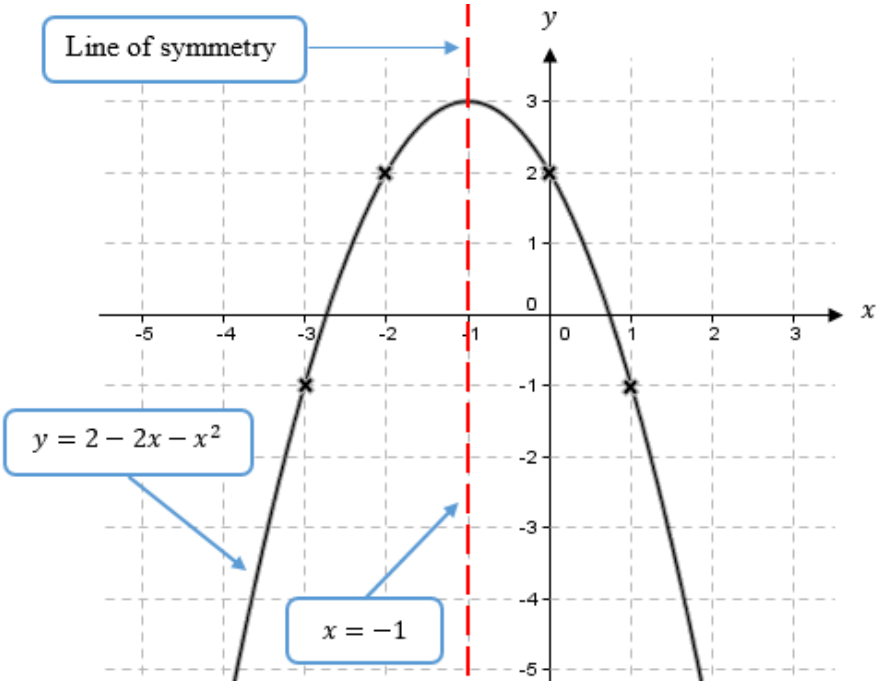


Fig 24.37

Do this yourself

Draw the line of symmetry in the following graphs and write the equation of the line of symmetry.

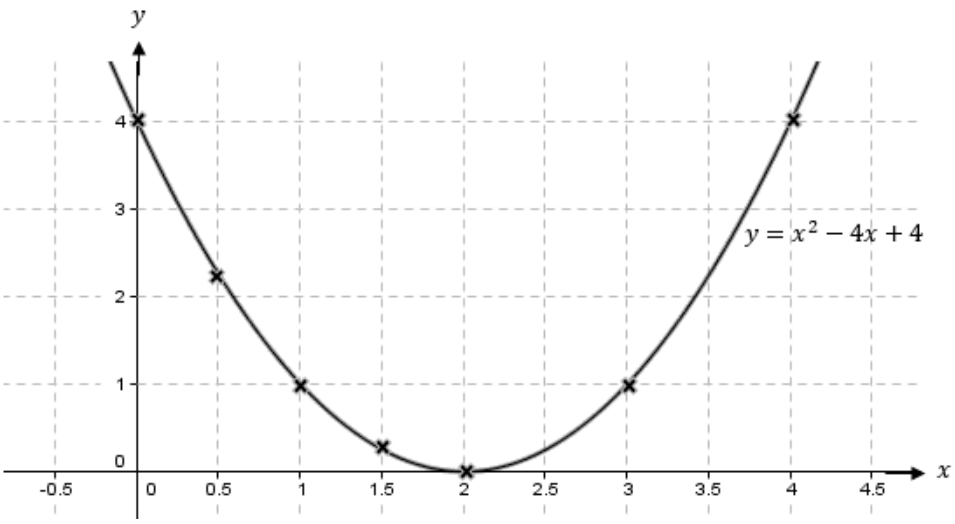


Fig 24.38

Here is the graph showing the line of symmetry

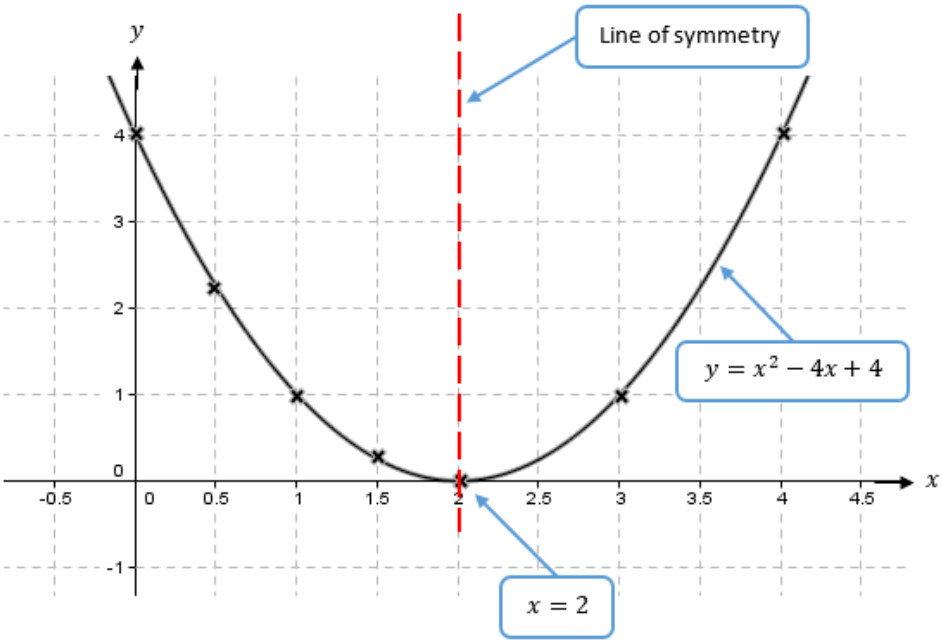


Fig 24.39

Activity (*24.2*) Quadratic function

Questions

1) (a) Copy and complete the table of values for the graph of the function $y = x^2 + 5x - 7$

x	–6	–5	–4	–3	–2	–1	0	1	2
y	–1	m	n	–13	–13	p	–7	q	7

(b) Using a scale of 2cm to represent 1 unit on the x–axis and 2cm to represent 5 units on the y–axis, draw the graph of $y = x^2 + 5x - 7$

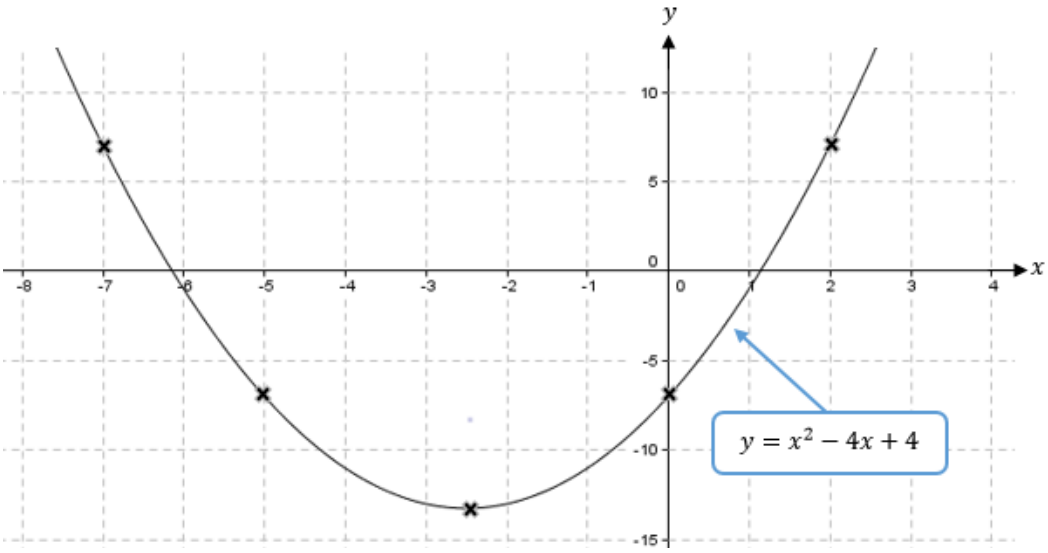


Fig 24.40

(c) Use your graph to answer the following questions

i. State the coordinates of the turning points

ii. Does the graph have a maximum or minimum turning point?

iii. Find the roots of the equation $x^2 + 5x - 7 = 0$

iv. Find the roots of the equation $x^2 + 5x - 7 = x$

v. Find the gradient of the curve at $x = -4$

vi. Draw the equation of the line of symmetry

(d) Find the area bounded by the curve, x–axis, $x = -5$ and $x = -3$

2) (a) Complete the table of values for $y = 3 - 2x - x^2$ below

x	–4	–3	–2	–1	0	1	2
y							

(b) Plot the graph of $y = 3 - 2x - x^2$

(c) Using your graph

i. Draw the line of symmetry and write the equation of the line of symmetry

ii. Find the gradient at $x = 0$

(d) Write down the roots of the equation $3 - 2x - x^2 = x + 2$

(e) Calculate the area bounded by the curve, the x- axis, y-axis and the line $x = -2$

Answers

1) (a) m = –7 n = –11 p = –11 q = –1

(b)

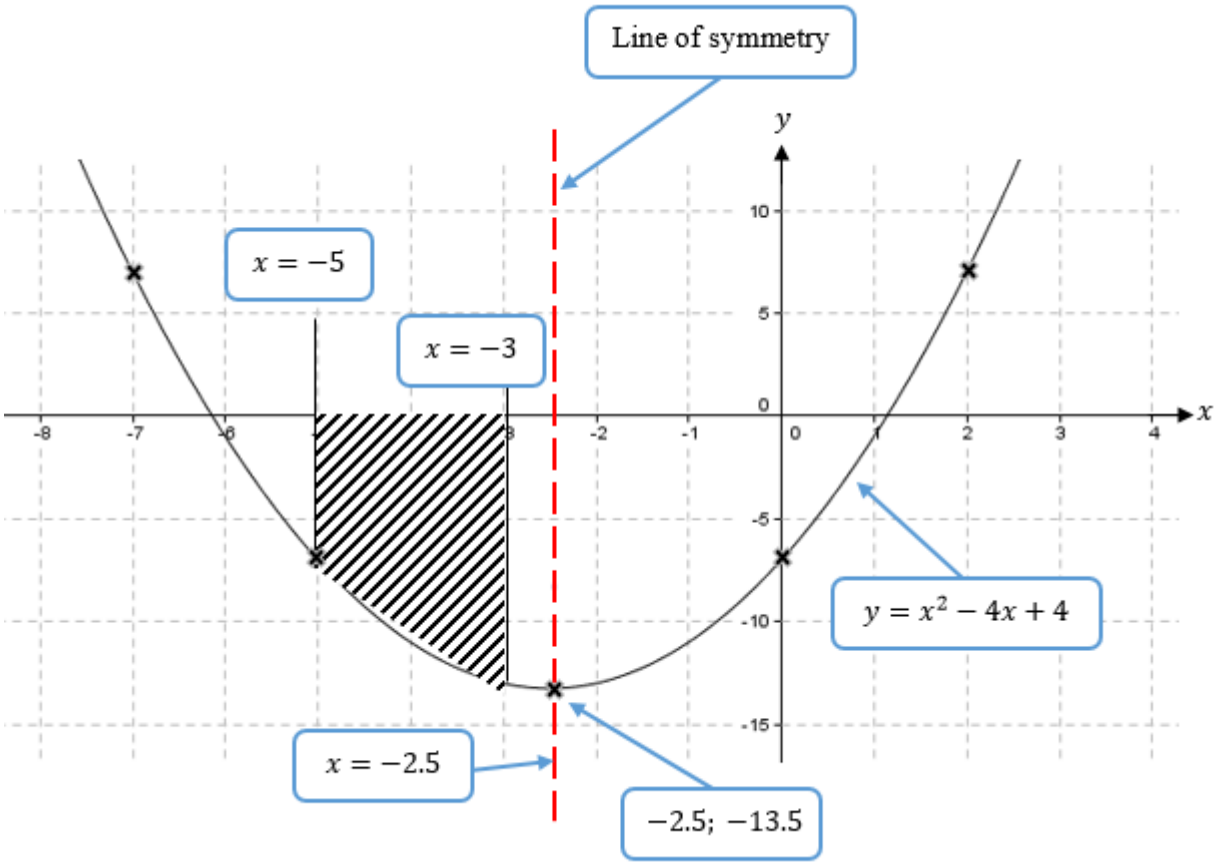


Fig 24.41

(c) i) (–2.5 ; –13.5) ii) minimum iii) –6.2 or 1.2

v) x = 0 (0ne root) v) –4 vi) On the graph

(d) 20 units²

2) (a)

x	–4	–3	–2	–1	0	1	2
y	–5	0	3	4	3	0	–5

(b)

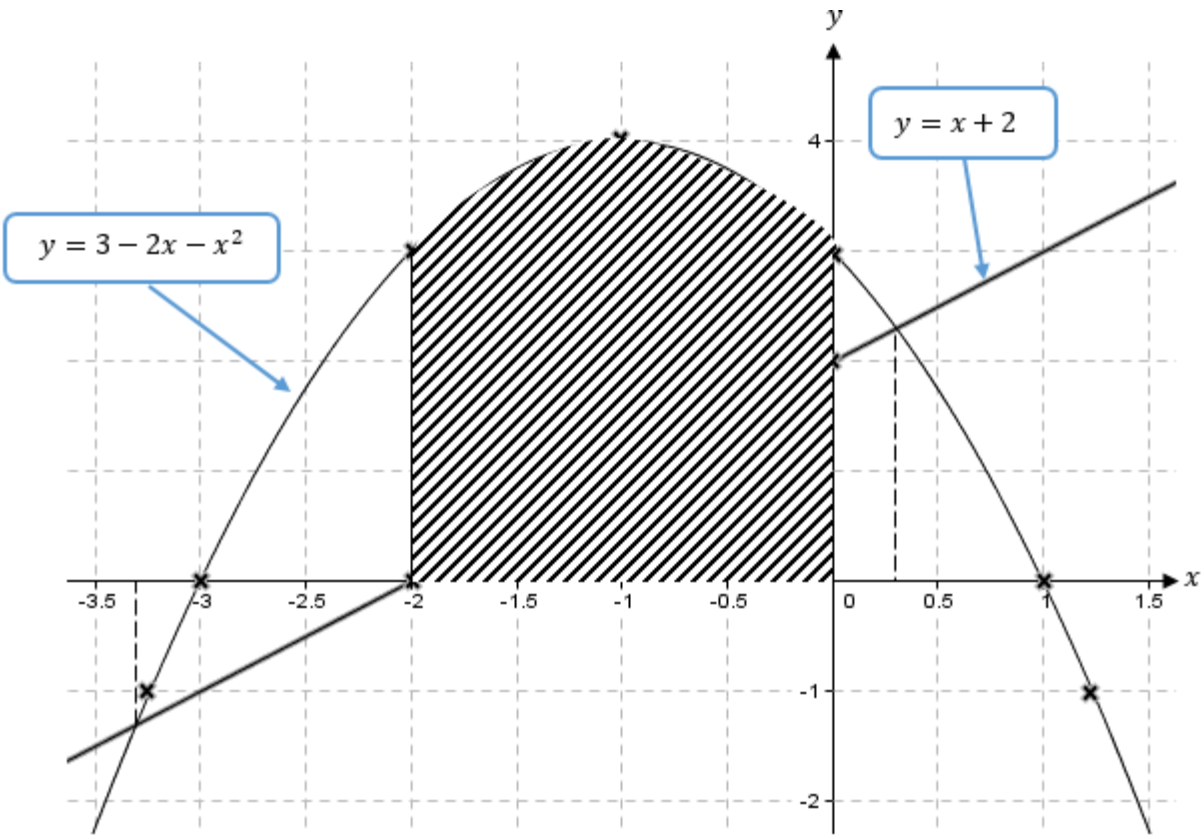


Fig 24.42

(c) i) $x = -1$ ii) –1 iii) –3,3 or 0,3

(d) 7 units²

24.3 INVERSE GRAPH/FUNCTION

What is an inverse function? Well if you do not have an answer to this question do not worry yourself. An inverse function is one in which the variable x is in the denominator. It can also be called a reciprocal function.

One main feature of an inverse function is that graph produces two separate curves.

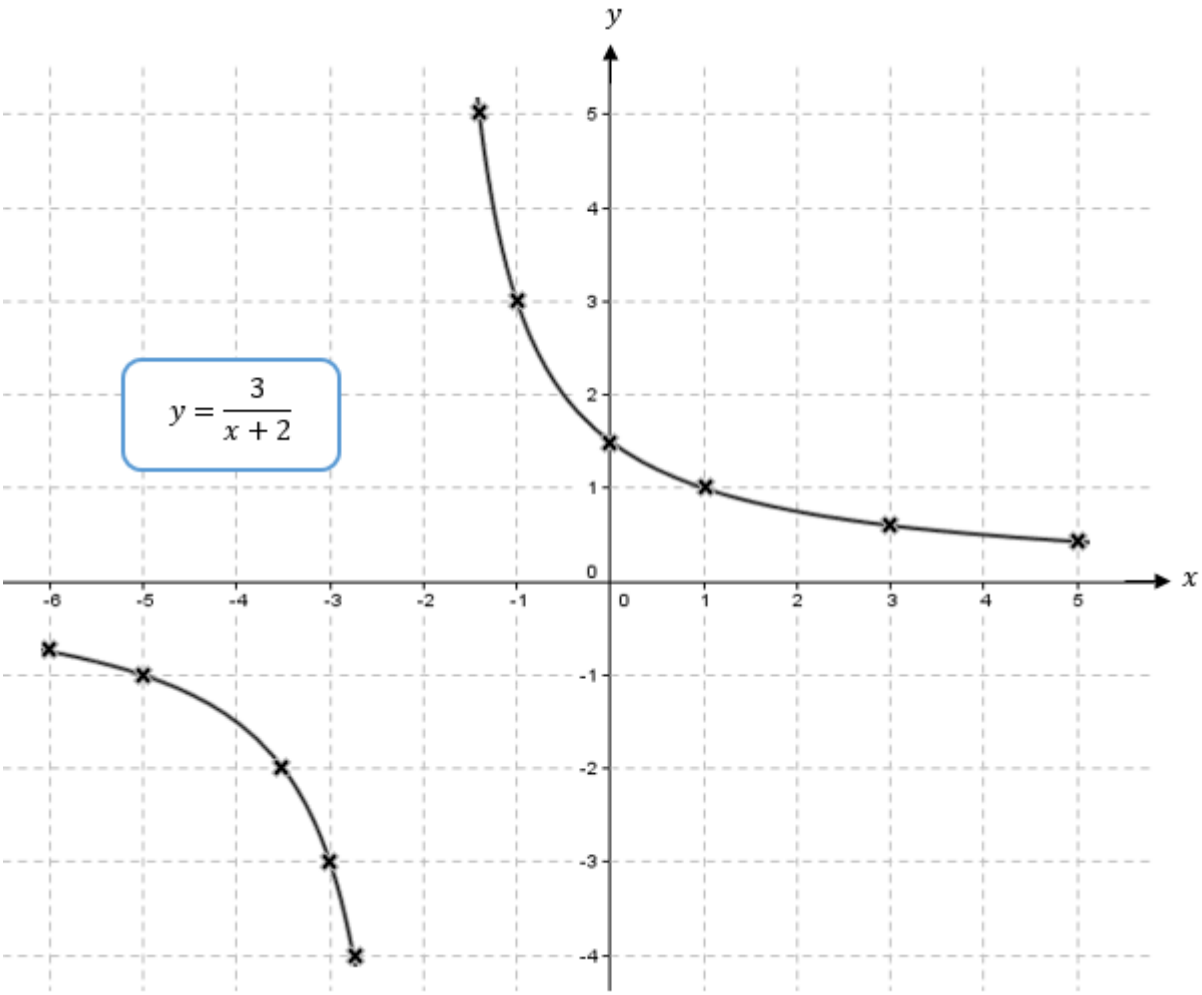


Fig 24.43

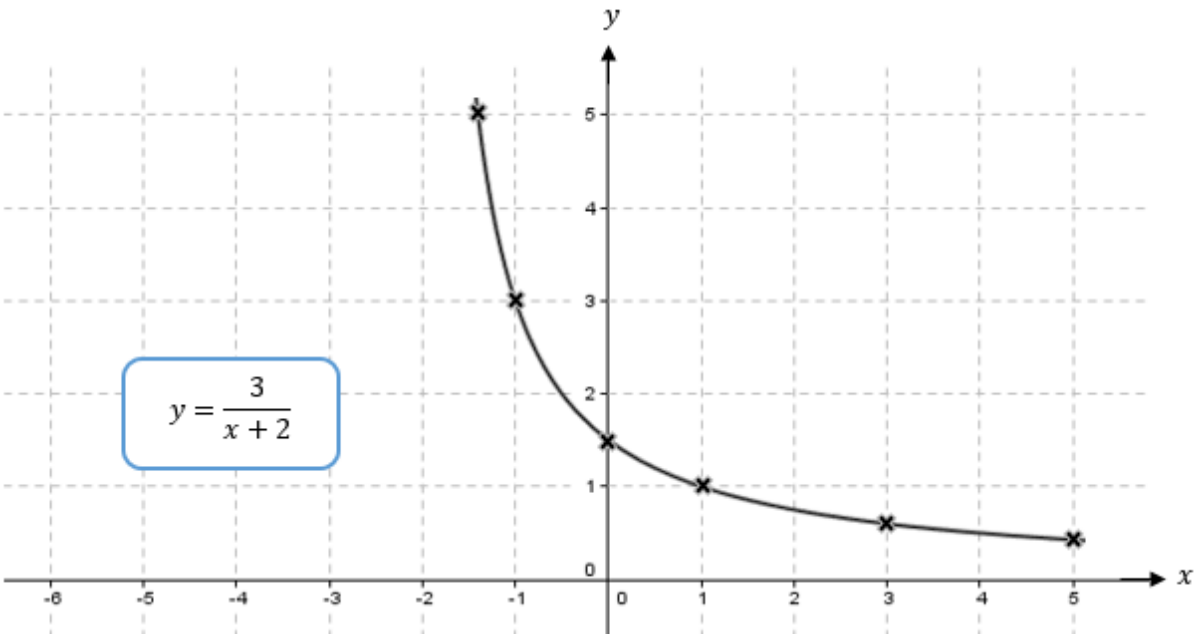


Fig 24.44

At times if you are given limits on the y or x–axis, only one part of the two curves is drawn. (it is still an inverse function

24.3.1 Plotting an Inverse function

Do you know how to plot a Inverse curve? Well we have been plotting curves all along when we were doing Quadratic graphs. The concept is still the same;

Step 1 – Making a table of values for a given Inverse function

Step 2 – Plotting the points on the Cartesian plane

Step 3 – Joining the points with a smooth curve

Let us consider the following example on how to plot an inverse function

Worked Example [12]

Questions

a) Copy and complete the table of values for $y = \frac{2}{x+1}$

x	–5	–4	–3	–2	$-1\frac{1}{2}$	$\frac{-1}{2}$	0	1	2	3	4	5
y	$-\frac{1}{2}$	–0,7	a	–2	–4	4	2	1	b	0,5	0,4	0,3

b) Using a scale of 2 cm to represent 1 unit on both axis, draw the graph of $y = \frac{2}{x+1}$ for $-5 \leq x \leq 5$

Solutions

a) Finding the value of "a" and "b"

Finding the value of "a"

When $x = -3$ you have to substitute in the equation to find the y value which is the value of 'a'

$$y = \frac{2}{-3+1}$$

$$y = \frac{2}{-2}$$

$y = -1$ therefore the value of $a = -1$

Finding the value of "b"

When $x = 2$ work this out in the space provided below

Finally our table of values is as follows

x	–5	–4	–3	–2	$-1\frac{1}{2}$	$\frac{-1}{2}$	0	1	2	3	4	5
y	$-\frac{1}{2}$	–0,7	a= –1	–2	–4	4	2	1	b= $\frac{2}{3}$	0,5	0,4	0,3

b) Now the table of values is used to plot points on the Cartesian plane. Join the points with a smooth curve (you should produce two separate curves

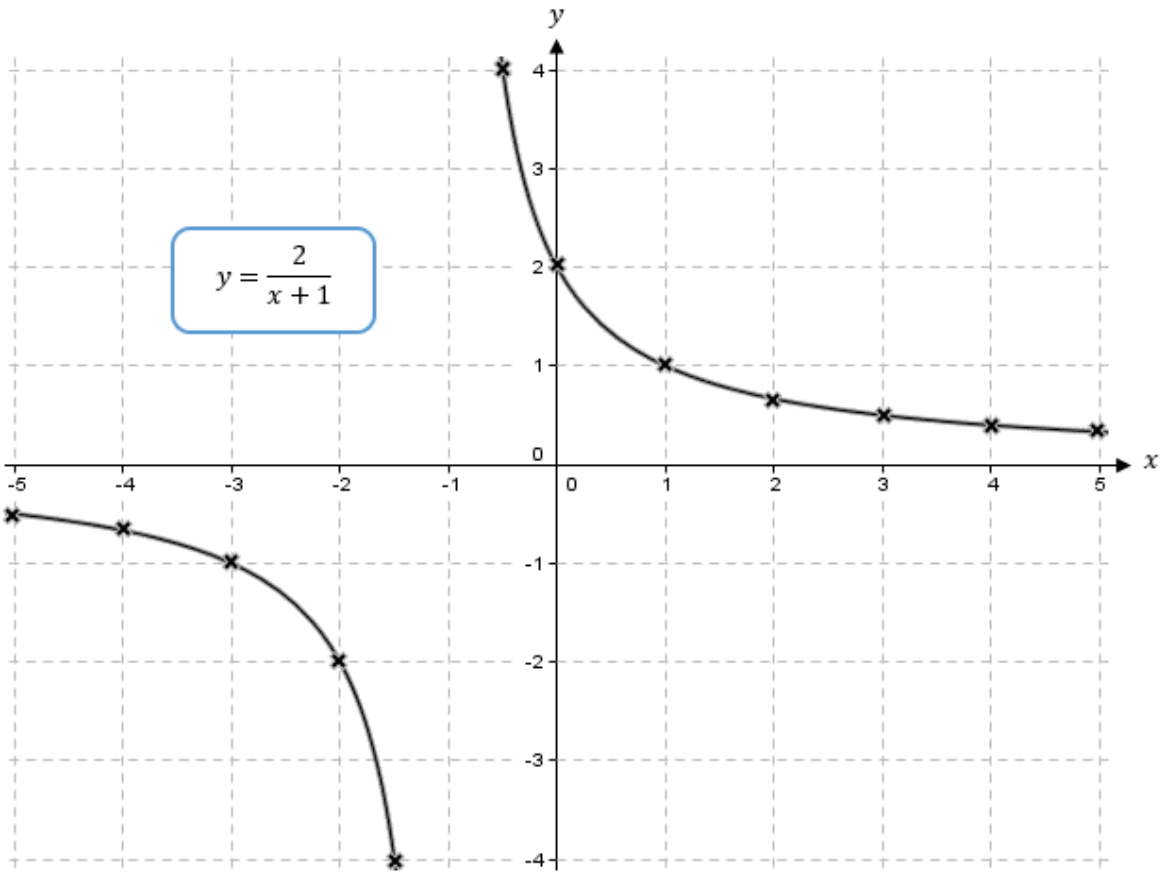


Fig 24.45

Do this yourself

The following is a table of values for $y = \frac{1}{x}$

x	–5	–4	–3	–2	–1	1	2	3	4	5
y	$\frac{1}{-5}$= –0.2	$\frac{1}{-4}$= –0.25	$\frac{1}{-3}$= –0.33	$\frac{1}{-2}$= –0.5	–1	1	$\frac{1}{2}$=0.5	$\frac{1}{3}$=0.33	$\frac{1}{4}$=0.25	$\frac{1}{5}$=0.2

On the blank graph page below, draw the graph of $y = \frac{1}{x}$

Plot the coordinates, using figures from the table of values above. After plotting points, join the points with a smooth curve, two separate graphs should be seen.

Fig 24.46

24.3.2 Area under the Inverse function/curve/graph

Do you still remember how we calculated the area under a quadratic function? If you have forgotten, visit "how to find area under a quadratic function ". Now that you remember how to calculate area under a quadratic function we now apply the same concept in finding the area under an inverse function'

We can either use the method of counting squares or the trapezium method (you have to remember that it is not always that you would find a trapezium, it might be a triangle or rectangle).

Let us now consider the following example on calculating the area under an Inverse function

Worked Example [13]

Questions

The following is the graph of $y = \frac{1}{x}$

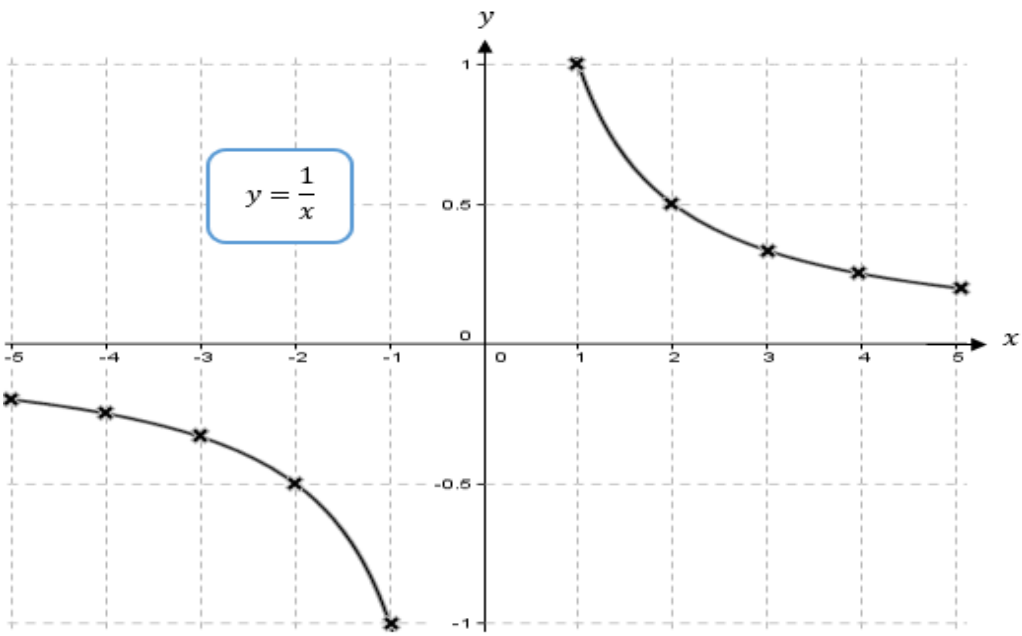


Fig 24.47

a) Find the area bounded by the curve, the x–axis, $x = 1$ and $x = 4$

Solutions

Show the area bounded by draw the lines $x = 1$ and $x = 4$ above. Your graph should look like the one below

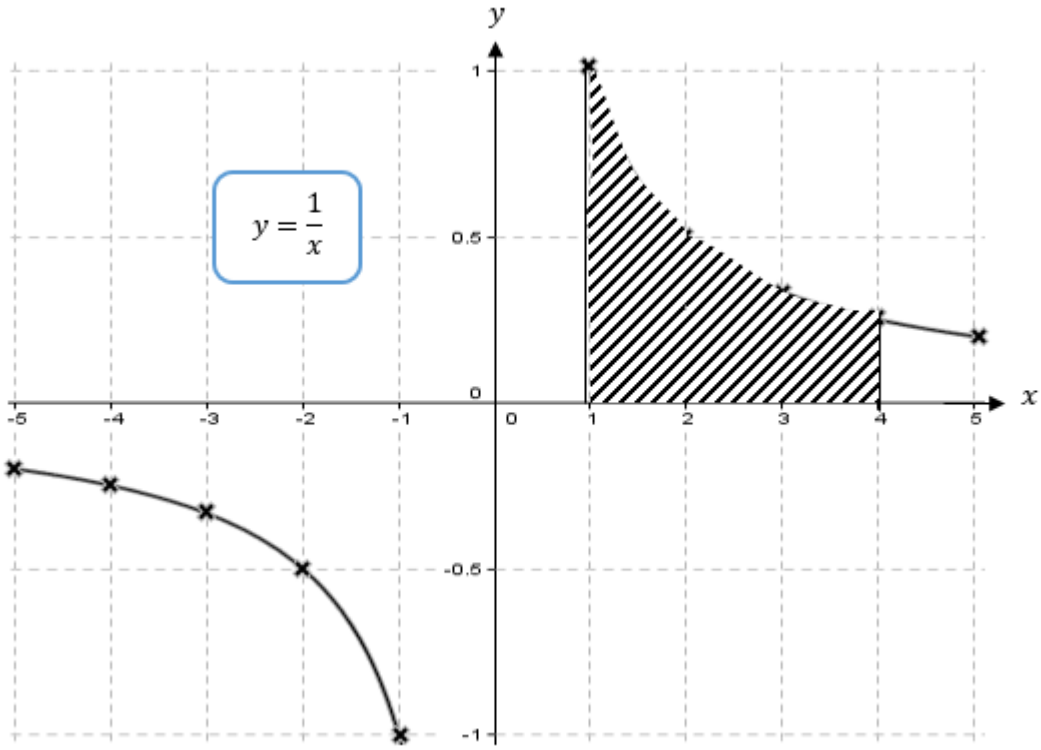


Fig 24.48

Decide on the method to use. Either the method of counting squares or the trapezium method.

Counting squares	Trapezium method
Approximately there 500 squares $0.1 \times 0.025 \times 500 = 1.3$ units²	Divide the area into trapeziums. Area A gives $\frac{1}{2}(1 + 0.5)1 = 0.75$ Area B gives $\frac{1}{2}(0.5 + 0.3)1 = 0.4$ Area C gives $\frac{1}{2}(0.3 + 0.25)1 = 0.275$ Area is: $A + B + C = 0.75 + 0.4 + 0.275 =$ 1.4units²

Do this yourself

The following is the graph of y= $\frac{1}{x+1}$, find the area bounded by the curve, the x–axis, $x = -2$ and $x = -3$

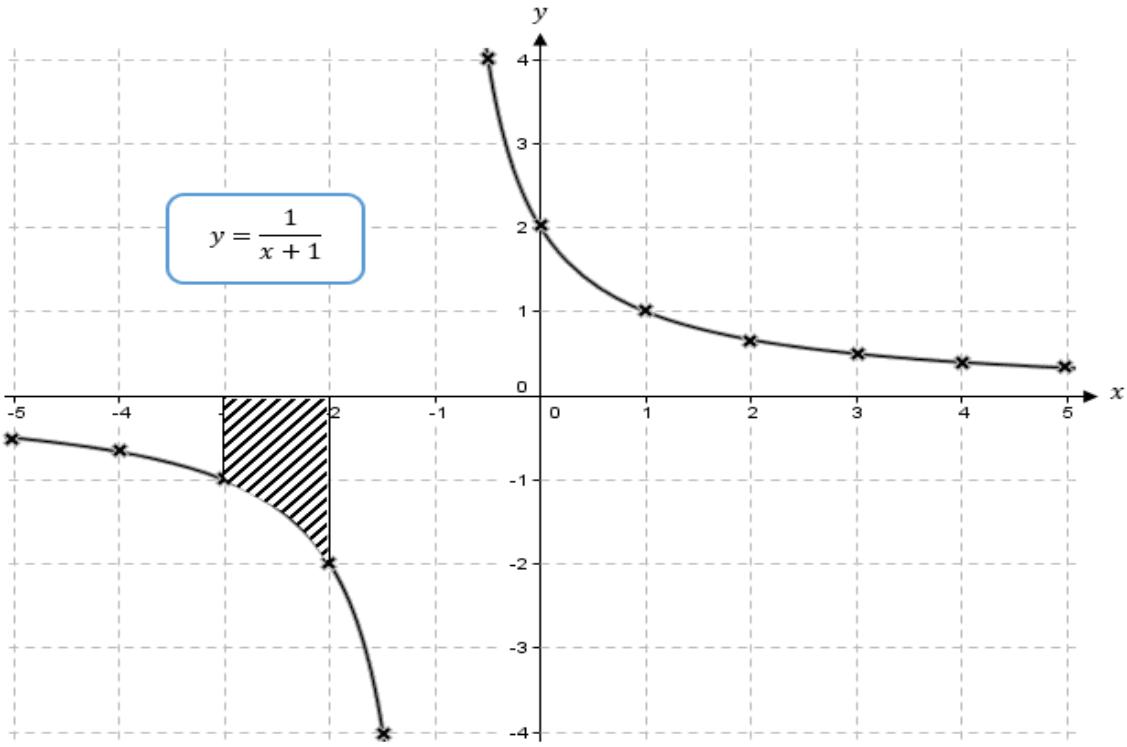


Fig 24.49

Use this space to calculate your area

24.3.3 Gradient of an Inverse function

Do you still remember how we calculated gradient of a curve at a particular point? Yes! We draw a tangent to the point and either

a) We find gradient by means of two coordinates

b) Or we use the right angled triangle method

Now let us consider an example on how we calculate the gradient of an Inverse function

Worked Example [14]

Question

The following is the graph of $y = \frac{2}{x+1}$, calculate the gradient at $x = 0$

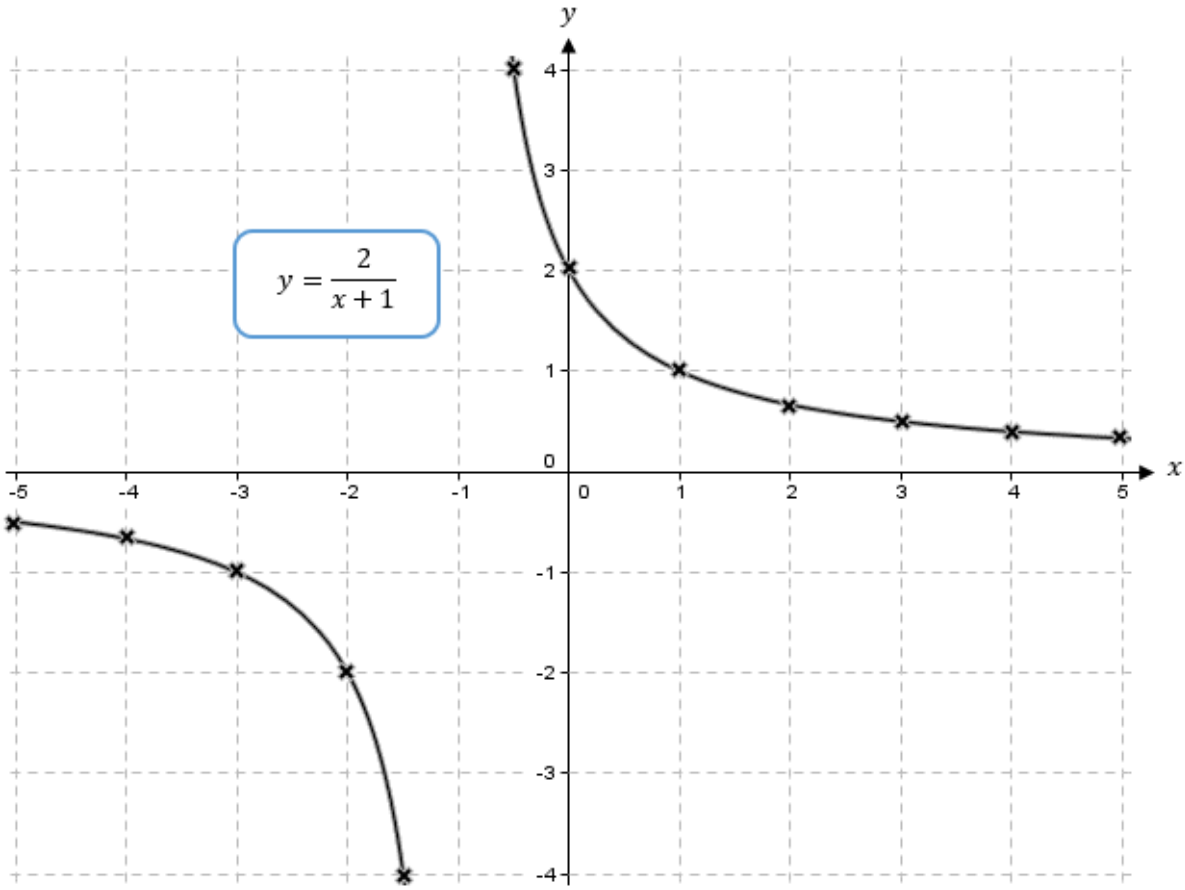


Fig 24.50

Solution

On the graph above, draw the tangent at $x = 0$ as shown below

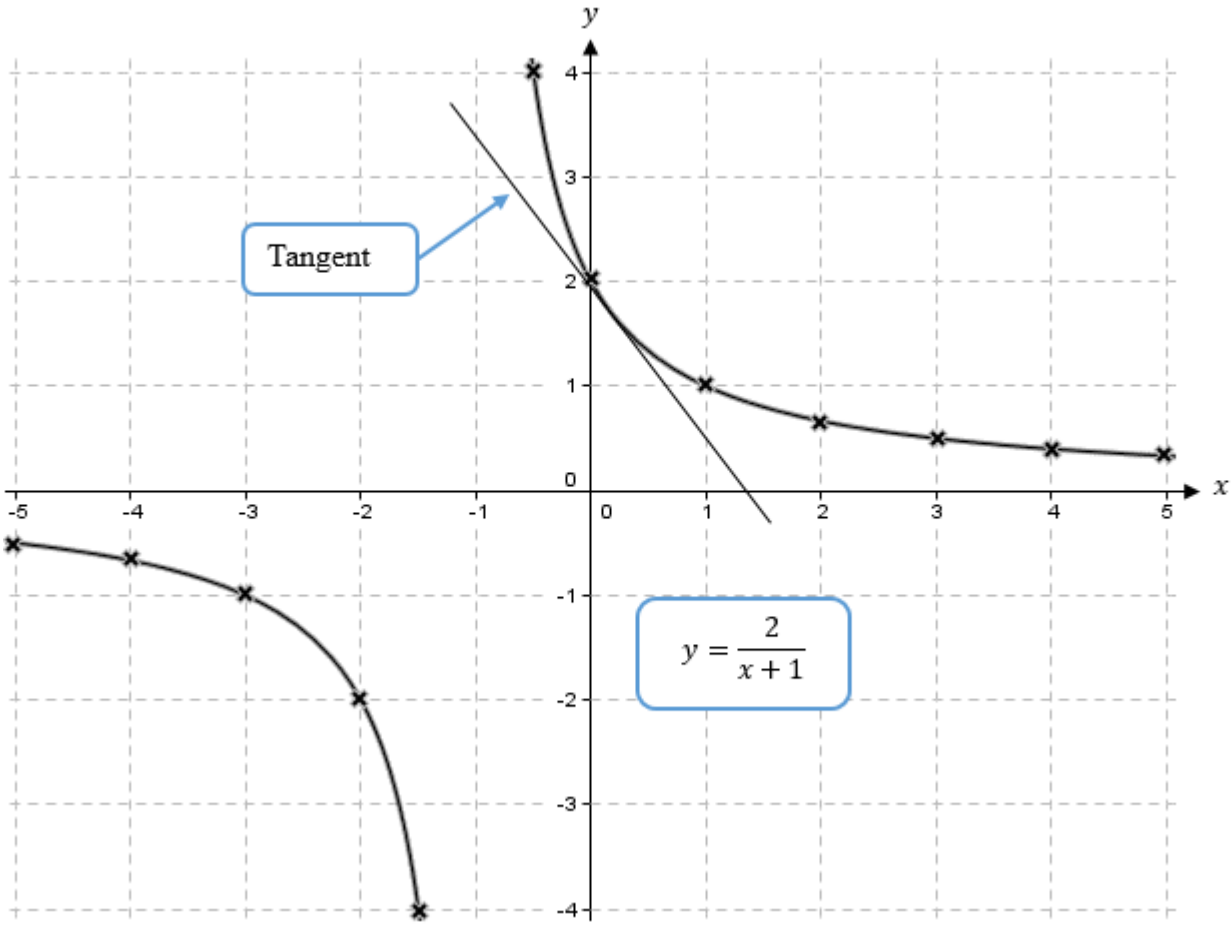


Fig 24.51

After drawing the tangent you use the two different methods of calculating gradient

Using any 2 coordinates on the tangent	Using the right angled triangle method
Take any 2 coordinates lying on the tangent above (1; 0,5) and (0 ; 2) Gradient= $\frac{2-0,5}{0-1} = \frac{1,5}{-1} = -1,5$	Draw a right angled triangle See the right angled tringle on the graph above Gradient = $\frac{1}{0,8} = -1,25$

If you have noticed the answers are different with each method used, you don't have to worry about that the answers are approximations. The answers just have to be approximately close to each other.

24.3.4 Intersection of a line and the Inverse function

Do you still remember what you find if a line and a curve intersect? Yes! We find the roots.

Now let us consider an example in which we will find the roots on an inverse function.

Worked Example [15]

Questions

The following is the graph of $y = \frac{1}{x}$

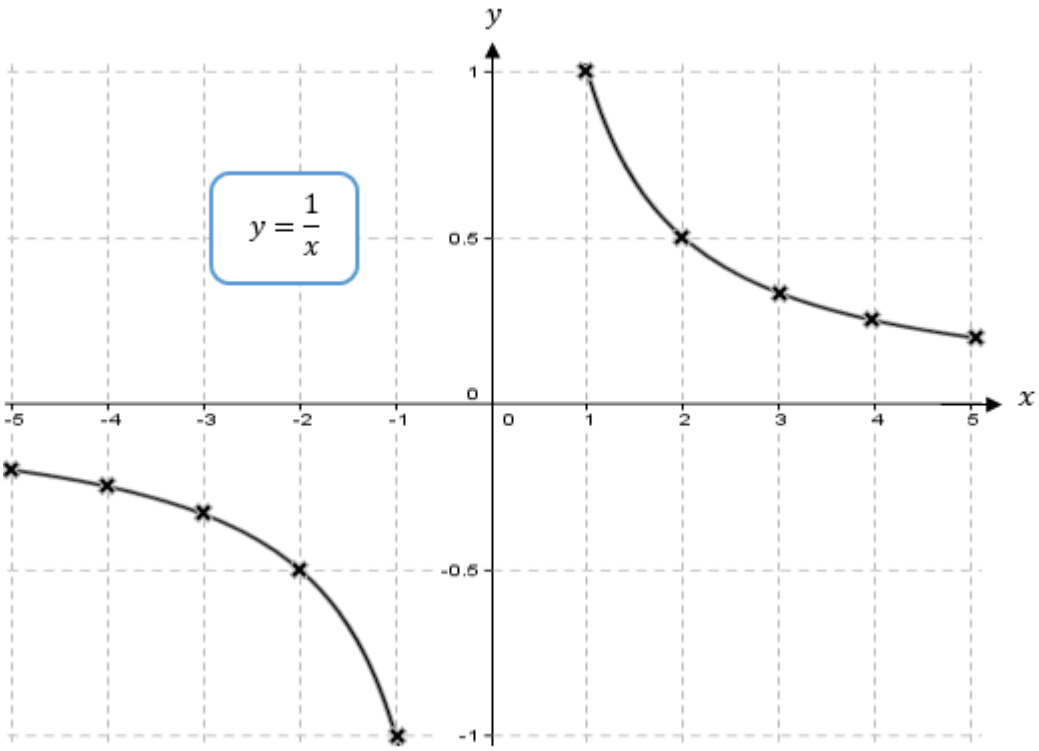


Fig 24.52

a) Find the roots for the equation $0.5 = \frac{1}{x}$

b) Solve the equation $\frac{1}{x} = 0.5x$

Solutions

a) To find the roots of the equation, you first write the original equation and subtract the one in question

$$\begin{array}{rl} & y = \frac{1}{x} \\ - & 0.5 = \frac{1}{x} \\ \hline & y - 0.5 = 0 \\ \hline & y = 0.5 \end{array}$$

Then you plot the result ($y = 0.5$) on the same graph above as shown below. At the point of intersection thus where you find the roots of the equation

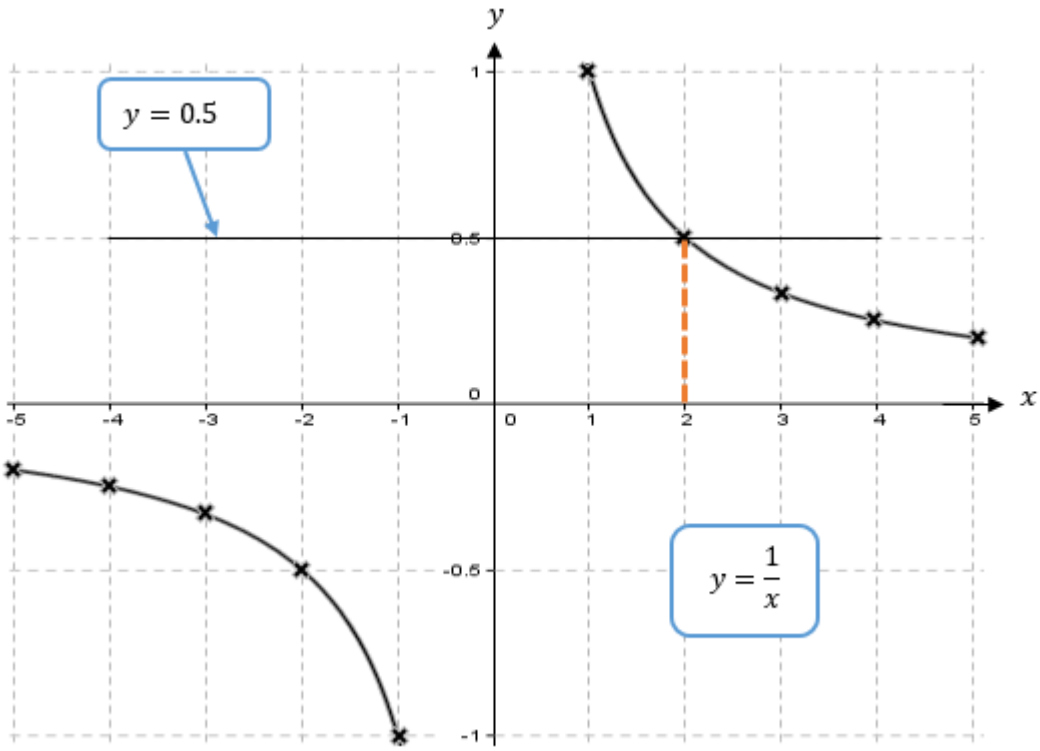


Fig 24.53

The root is 2

b) We subtract the equation from the original equation

$$\begin{array}{rl} & y = \frac{1}{x} \\ - & 0.5x = \frac{1}{x} \\ \hline & y - 0.5x = 0 \\ \hline & y = 0.5x \end{array}$$

Make y subject of the formula

Now come up with a table of values so that you will be able to plot the graph of $y = 0.5x$ on the same axis as with $y = \frac{1}{x}$ above as shown below.

x	–1	1	2
y	–0.5	0,5	1

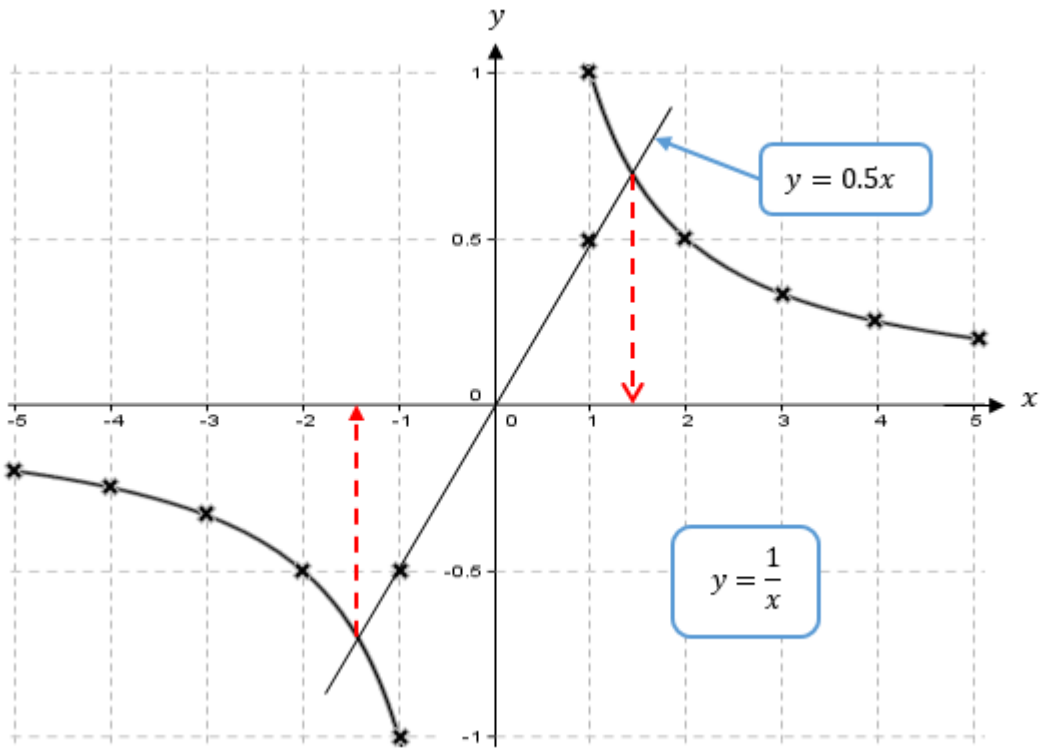


Fig 24.54

The roots are taken on the x–axis where the two graphs intersect as shown on the graph above and the roots are –1.4 or 1.4

Activity (*24.3*) Inverse function

Questions

1) (a) The following is an incomplete table of values for $y = \frac{3}{x+2}$

x	–6	–5	–4	–3	–2,5	–1	0	1	2
y	$\frac{-3}{4} = -0.75$	p	$-1\frac{1}{2}$=–1,5	q	–6	r	1,5	1	$\frac{3}{4}$=0,75

i. Calculate the value of p, q and r

ii. Using a scale of 2cm to represent 1 unit on both axis, draw the graph of $y = \frac{3}{x+2}$ for $-6 \leq x \leq 2$

(b) Use your graph to

i. find the gradient at x = 1

ii. find the area enclosed by the curve, x–axis, $x = -5$ and $x = -3$

iii. find the roots of the equation $x + 1 = \frac{3}{x+2}$

2) (a) Complete the table of values of $y = \frac{4}{x^2}$

x	1	1,5	2	2.5	3	4	5
y	4		1			$\frac{1}{4}$	

(b) Using a scale of 2cm to represent 1 unit on both axis, draw the graph of $y = \frac{4}{x^2}$

(c) Using your graph

i. Find the gradient at $x = 3$

ii. Calculate the area bounded by the curve, $x = 2$ and $x = 4$

iii. Find the roots of the equation $x = \frac{4}{x^2}$

24.4 Summary

In this unit we have about how to interpret the functional notation, construct a table of values from a given function, use given scale correctly to construct the functions, draw graphs (functions) of the form

- $y = ax^2 + bx + c$ (quadratic function)
- $y = \frac{1}{ax+b}$ (inverse function)

We also have learnt how to estimate the area under the curve, draw tangents at given points, estimate the gradient of lines and curves at given points, solve equations graphically and use graphs to determine the maximum and minimum turning points.

24.5 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

24.6 Assessment Test

1) The following is the table of values for the graph of $y = 7 - 5x - x^2$

x	–6	–5	–4	–3	–2	–1	0	1	2
y	1	7	11	a	13	11	b	1	–7

(a) find the values of a and b (2)

(b) using the scale of 2cm to represent 1 unit on the horizontal axis and 2cm to represent 5 units on the vertical axis, draw the graph of $y = 7 - 5x - x^2$ for $-7 \leq x \leq 2$ (4)

(c) use your graph to answer the following

I. state the coordinates of the maximum point (1)

II. solve the equation $7 - 5x - x^2 = 1$ (3)

III. find the gradient of the curve at the point where $x = 0$ (2)

2) (a) The following is an incomplete table of values for the function $y = \frac{3}{x+2}$

X	–6	–5	–4	–3	–2.5	–1	0	1	2
Y	–0.75	–1	–1.5	p	–6	3	1.5	q	0.75

i. Calculate the value of p and q (2)

ii. Using a scale of 2cm to represent 1 unit on both axes draw the graph of $y = \frac{3}{x+2}$ for $-6 \leq x \leq 2$ (4)

(b) using you graph,

i. find the gradient of the curve at the point where $x = 1$ (2)

ii. solve the equation $2x + 3 = \frac{3}{x+2}$ (4)

UNIT 25 - GEOMETRY 5

CONTENTS

25.1 INTRODUCTION

This unit is a continuation of Unit 11 on Geometry. In this unit we are going to carry on with Geometric construction where we are going to look at construction of quadrilaterals, construction of a perpendicular from a point to a line, construction of an inscribe circle of a triangle and construction of a circumscribe circle of a triangle as well. We are also going to look at Loci.

Objectives

After going through the unit, you should be able to:

- ✓ construct a circumscribed circle of the triangle
- ✓ construct a inscribed circle of a triangle
- ✓ applying the laws of locus
- ✓ present different construction concept on a single diagram
- ✓ apply construction concepts on bearing

Key words

The following are key words and their meaning in Mathematics

Construction - is the drawing of shapes, angles and lines accurately using geometrical or mathematical instruments

Bisection - this is to divide into two equal pieces or parts

Locus - is a set of points satisfying a given rule

Vertex - the point of intersection of any lines on a given sharp. It might be at times called corners of a shape.

Equidistant - the word is made up of two words, equal and distance therefore the word means being at an equal distant from the same point or thing.

Supplementary angles – these are two angles which add to 180°.

Time

You need to be done with this unit in 10 hours.

Study skills

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems in Geometrical Construction and Loci as possible for you to grasp all the concepts in this topic. Make sure that your geometrical instruments are tightened and the pencil is sharpened for accuracy.

25.2 GEOMETRICAL CONSTRUCTION

25.2.1 Construction of Quadrilaterals

In this section we are going to learn to construct different types of quadrilaterals which are;

Rectangles, parallelograms, trapeziums, rhombus and kites

25.2.1.1 Construction of a rectangle

Look back

Draw a sketch of a rectangle in the space below

List the properties of a rectangle

.....

.....

.....

Well if you have managed to draw the rectangle you should observe that its opposites sides are equal.

You should also note that the angles are all 90°angles. Now let's do the following example to construct a rectangle accurately.

Worked Example [1]

Question

Construct a rectangle ABCD with AB=6cm and BC =4cm

Solution

Tip; angles are not given, it is already known that the angles of a rectangle are all 90° angles.

Step1 - Open a radius of 6 cm to construct the line AB=6cm, and make 90°at A and B

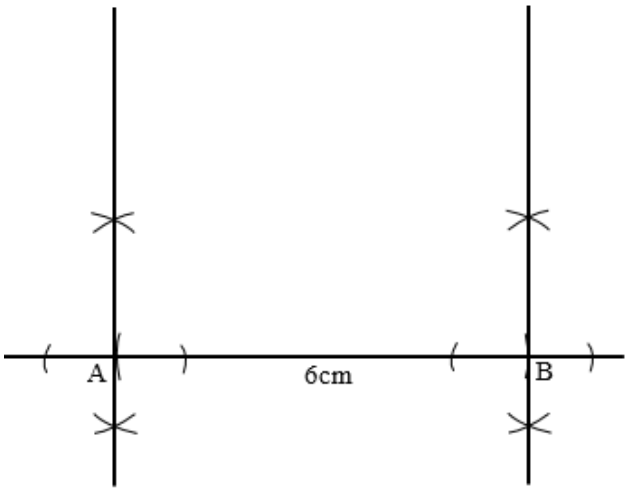


Fig 25.1

Step 2 - Open a radius of 4cm and make an arc above A and also above B and label the two arcs C and D

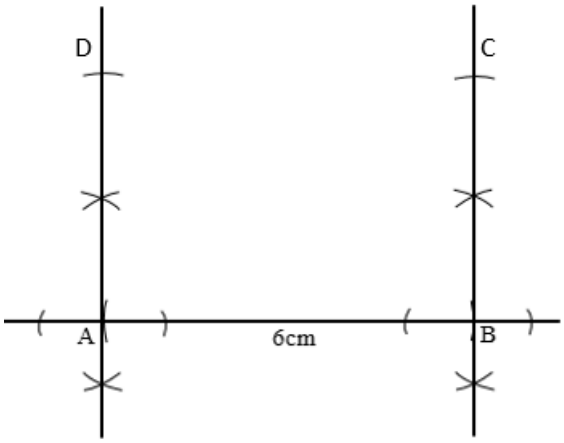


Fig 25.2

Step 3 - Join the point C and D with a straight line

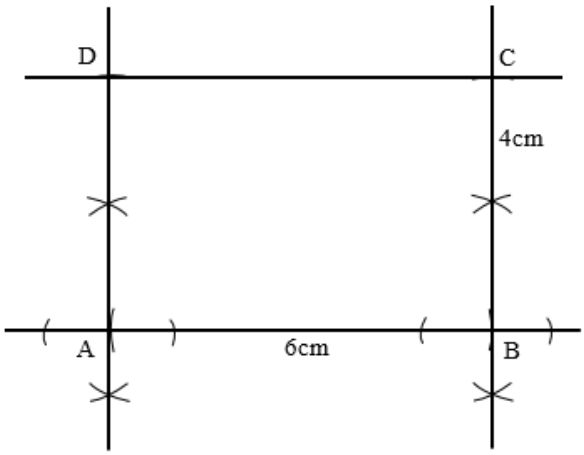


Fig 25.3

Points to remember;

Given the name of the polygon, you should apply your understanding shape properties.

It is not always that they give you all the dimensions and angles of a stated shape especially a rectangle

Practice

Construct a rectangle XYWZ with XY=9cm and YW=5cm

25.2.1.2 Construction of a parallelogram

Draw a sketch of a parallelogram in the space below

List the properties of a parallelogram

.....

.....

.....

If you got this correctly, you should find out that opposite sides of a parallelogram are equal

And adjacent angles are supplementary (angles which add to 180°)

Worked Example [2]

Questions

Construct a parallelogram ABCD with AB=6cm, ABC=120° and BC=4cm

Solution

Step 1 - Open a radius of 8cm and construct the line AB=6cm and construct a 120° angle at B

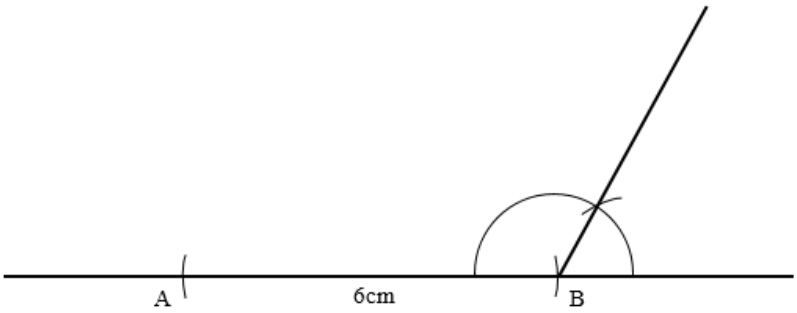


Fig 25.4

Step 2 - Construct the 60° angle at BAD

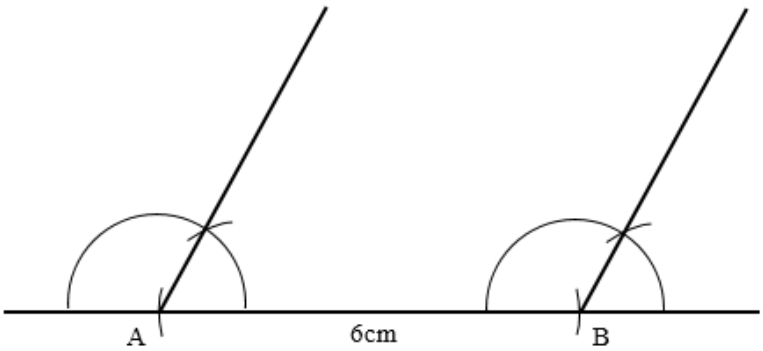


Fig 25.5

Step 3 - Open a radius of 4cm and make arcs with the needle point at A and B and label D and C respectively

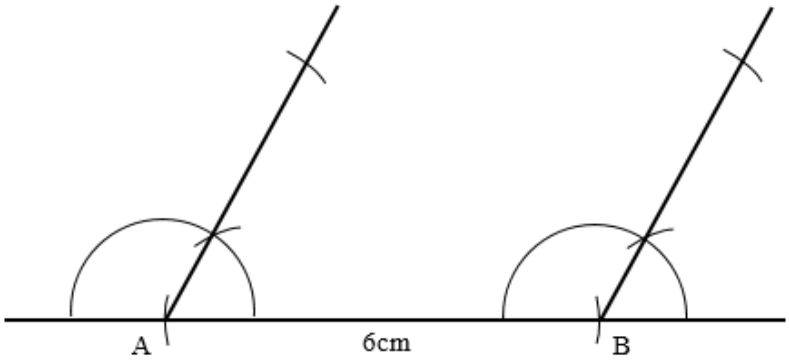


Fig 25.6

Step 4 - Join C and D with a straight line

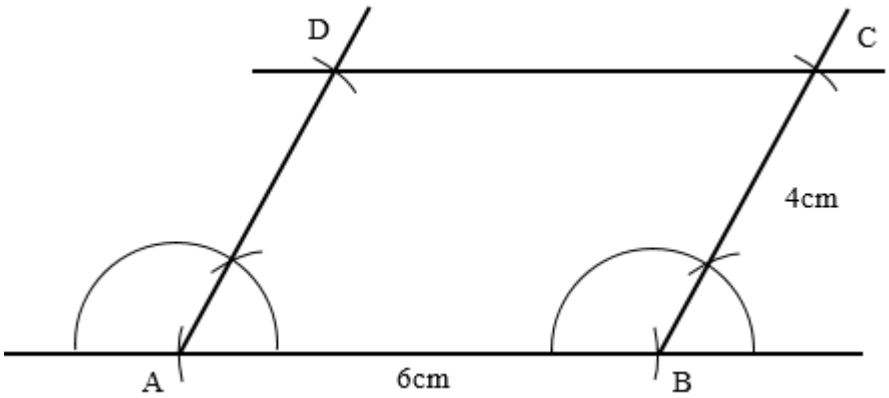


Fig 25.7

Activity (*25.1*) Constructing quadrilaterals

Questions

Using ruler and compass only;

1. Construct a parallelogram XYWZ with WX=8cm and XY=5cm and WXY=135°
2. Construct a rectangle ABCD with AB=6CM and BC=5CM
3. Construct a square XYWZ with sides 5CM

25.2.1.3 Construction of a trapezium

Looking back

Sketch a trapezium in the space below

List the properties of a trapezium

.....

.....

.....

If you have failed to do the above, don't worry yourself, go back to unit 7 and look at trapezium

Did you find out that they are two parallel sides on a trapezium?

Worked Example [3]

Question

Construct a trapezium ABCD, such that AB=6cm, ABC=60°, BAD=90° with AD=4cm and BC=5cm

Solution

Step 1- Construct line AB=6cm and angle ABC=60° at point B

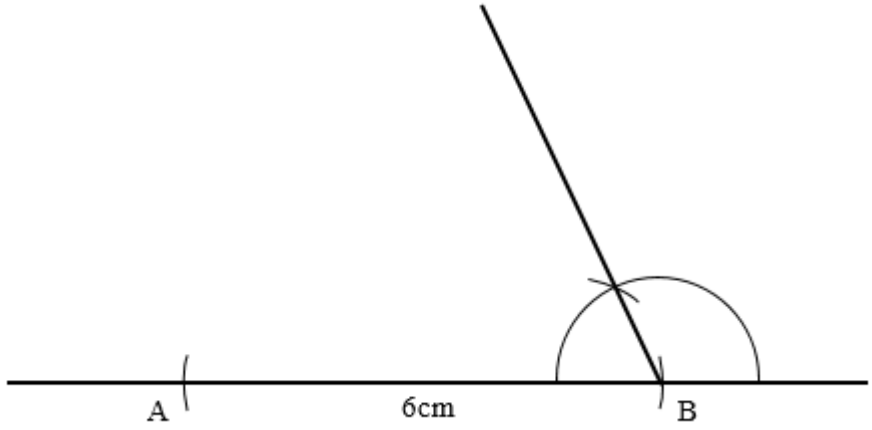


Fig 25.8

Step 2 - Open a radius of 5cm on your compass and make an arc and label the point of intersection of the arc and the line C

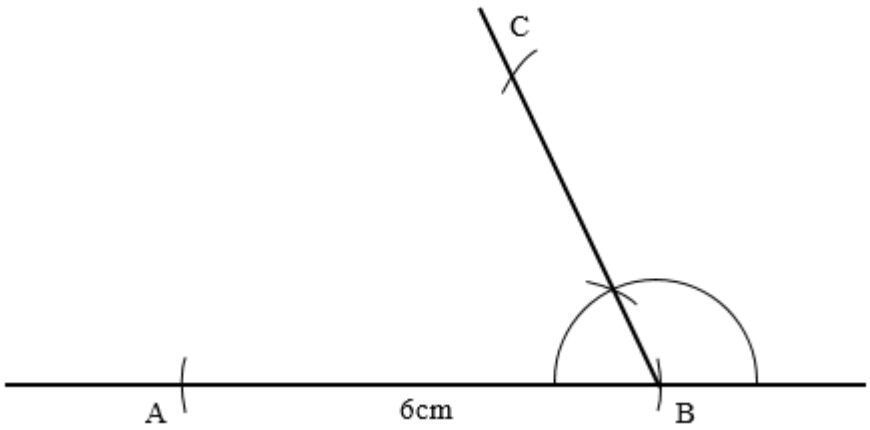


Fig 25.9

Step 3 - Construct BAD=90°at point A

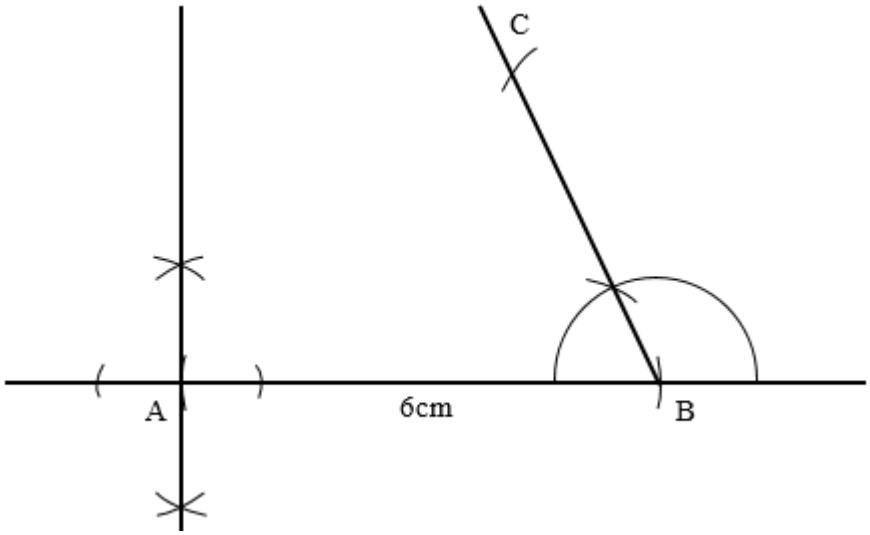


Fig 25.10

Step 4 - Open a radius of 4 cm on your compass to make an arc and label D, join D and C with a straight line

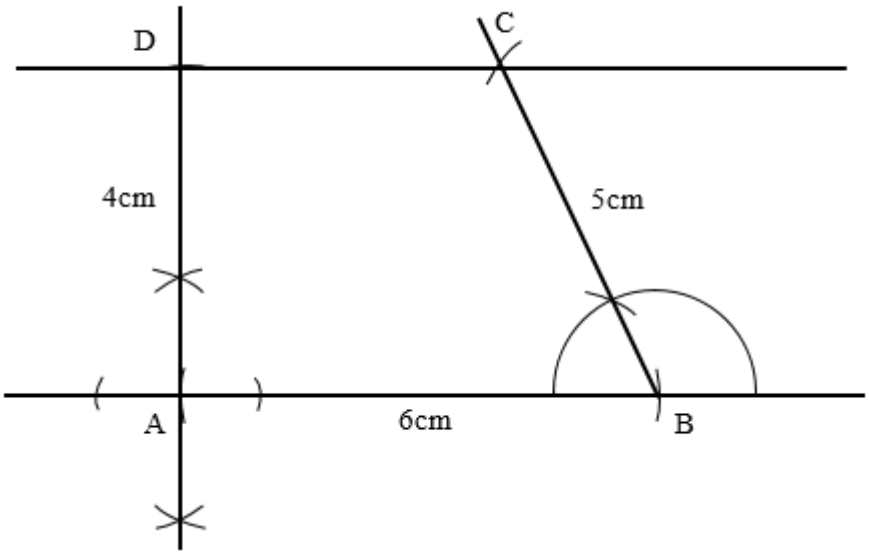


Fig 25.11

Practice

Construct a trapezium WXYZ with WX=9cm, WXY=90°, XY=5CM, XWZ=30°and WZ=7cm

25.2.1.4 Construction of kites

Looking back

Sketch a kite in the space below

List the properties of a kite

.....

.....

.....

If you have failed to draw a kite, don't trouble yourself, go back to Unit 7 and look at a kite and its properties

Did you find out that they are two equal shorter sides and two equal longer sides, Let's try out the example below to find out how a kite is constructed

Worked Example [4]

Question

Construct a kite ABCD with AB=4cm, ABC=120° and BC=–6cm

Solution

Step 1 - Open a radius of 4cm to make the line AB and make an angle of 120° at B

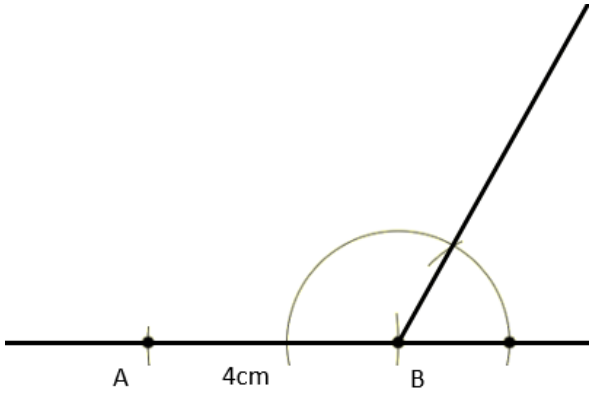


Fig 25.12

Step 2 - Open a radius of 6cm and with the needle point at B, make an arc and label the arc C

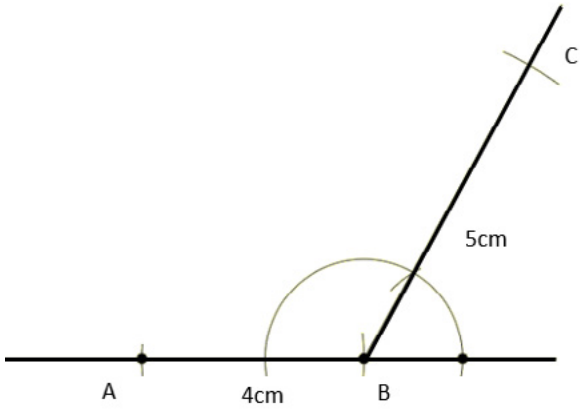


Fig 25.13

Step 3 - With the needle end at A, open a radius of 4cm and make an arc

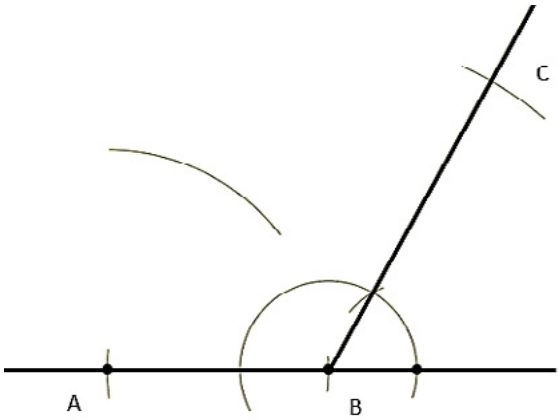


Fig 25.14

Step 4 - With the needle end at point C, open a radius of 5cm and make an arc to cross another arc, label the point of intersection of the arcs point D

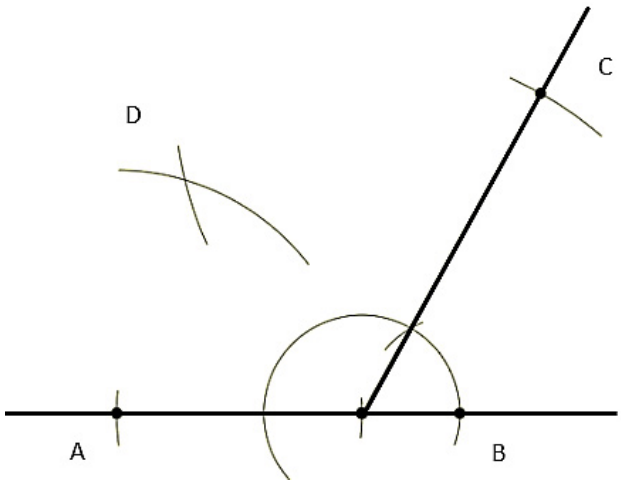


Fig 25.15

Step 5 - Join point A to point D and point D with point C thus completes the kite

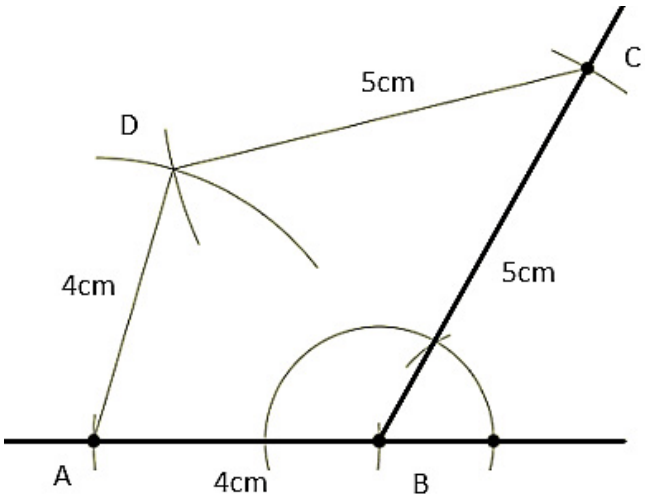


Fig 25.16

Practice

Construct a kite ABCD with AB=5cm, ABC=135° and BC=8cm

25.2.1.5 Construction of rhombus

Look back

Sketch a rhombus in the space below

Write down the properties of a rhombus

.....

.....

.....

If you have failed to do the above, go to Unit 7 and see a rhombus and its properties.

You should discover from unit 7 that a rhombus have all equal sides and adjacent angles are supplementary

Now let's look at how a rhombus is constructed in the following example

Worked Example [5]

Question

Construct a rhombus ABCD in which AB=4cm , ABC=120°

Solution

Step 1 - Construct line AB=4cm and at point B construct a 120°angle

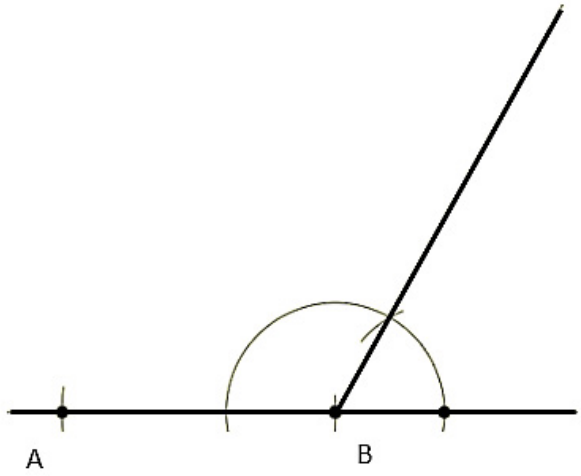


Fig 25.17

Step 2; construct a 60°angle at A

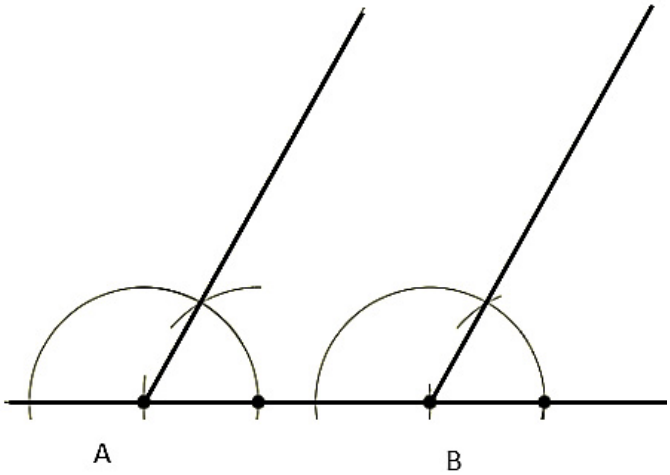


Fig 25.18

Step 3 - Open a radius of 5 cm and with the needle point at A, make an arc and with the needle point at B , make another arc . Label the intersection points C and D

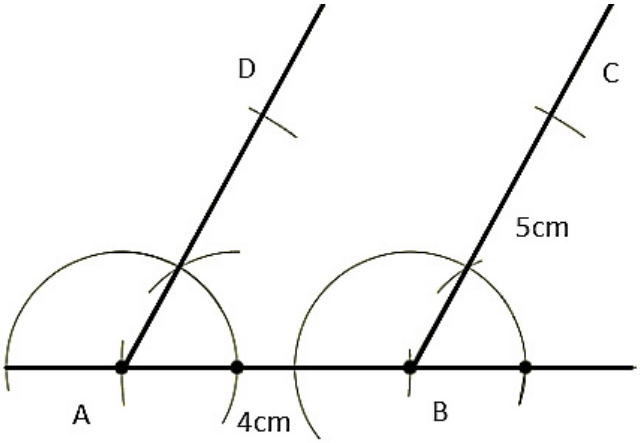


Fig 25.19

Step 4 - Join the point C and D with a straight line, to make the rhombus.

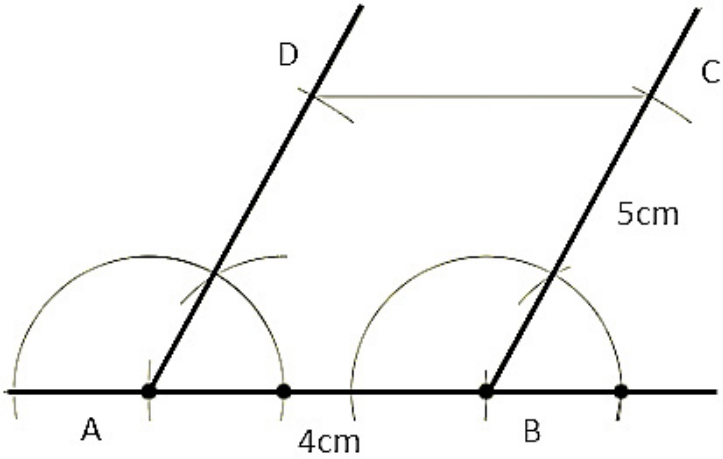


Fig 25.20

PRACTICE

Construct a rhombus XYWZ with XY=6,5cm and ZXY=30°

Activity (*25.2*) Constructing quadrilaterals

Questions

1. Construct a rhombus ABCD with AB=5 cm and ABC=135°
2. Construct a kite XYWZ with XY=6cm, YXZ=120° and XZ=4cm.
3. Construct a trapezium ABCD with AB=10cm, BC=4cm, ABC=90°, BAD=45°and AD=5cm
4. Construct a parallelogram CDEF , with CD=6.5cm , CDE=150° and DE=4cm

25.2.2 Construction of a perpendicular line from a point to a line

What do you know about perpendicular lines?

.....

.....

.....

.....

Yes these are lines which meet and make an angle of 90°. Perpendicular lines are lines which meet at a 90°angle. Now here is how we construct a perpendicular line from a point to a line

Worked Example [6]

Questions

a. Construct a triangle ABC with AB=5cm, ABC=120° and BC=6cm'

b. Construct a perpendicular line from C to AB produced

Solutions

Tip; the perpendicular line is the shortest distance from C to AB produced

Step 1 - Construct the triangle ABC (at this point you are now capable of constructing a triangle)

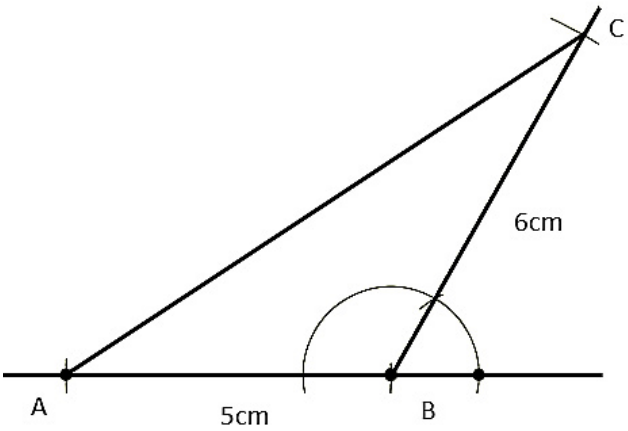


Fig 25.21

Step 2 - Produce line AB. (this means to extend the line AB)

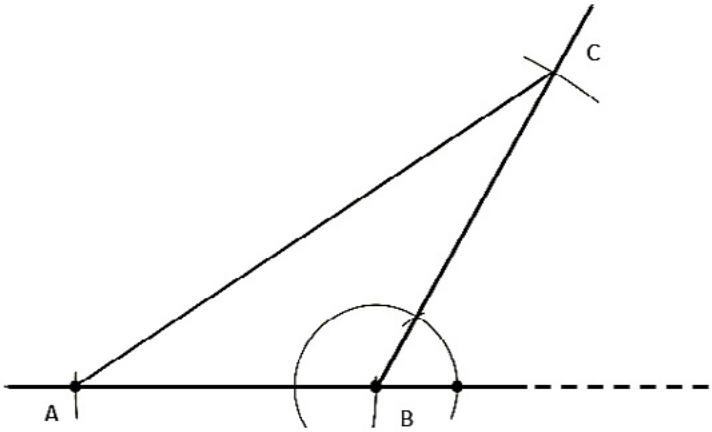


Fig 25.22

Step 3 - With the needle point at C, make an arc to cross the extended line

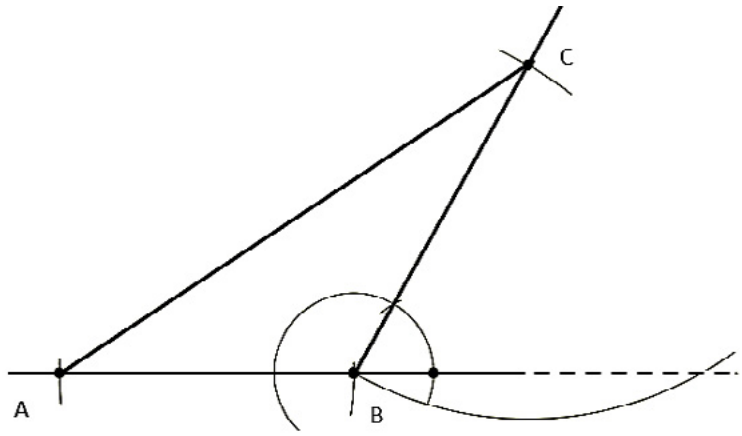


Fig 25.23

Step 4 - The points of intersection of the arc and the line AB are noted. With the needle point at the points of intersection, make arcs like the ones below

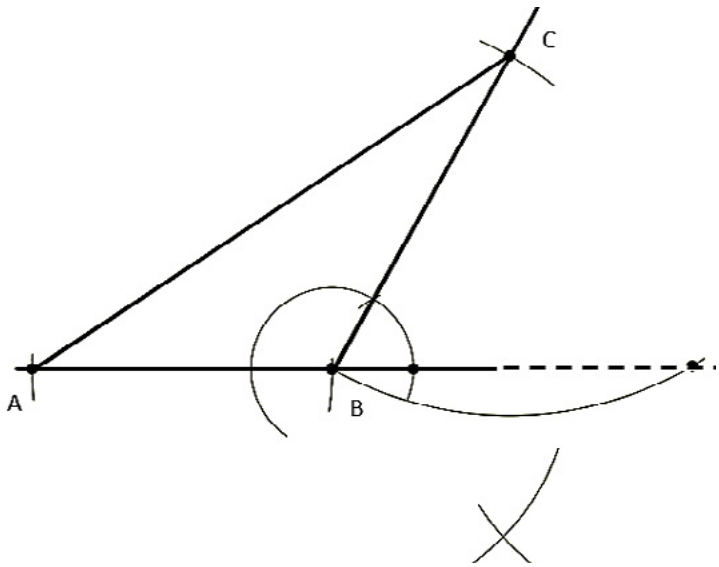


Fig 25.24

Step 6 - Now from point C join using a ruler to the point of intersection of the arcs.

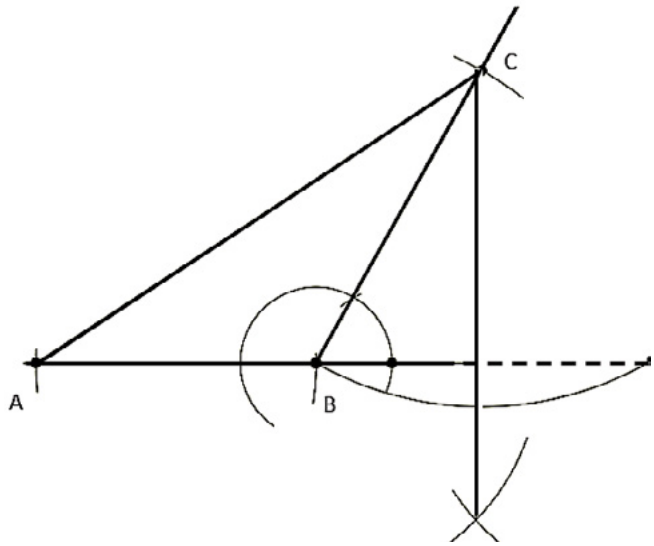


Fig 25.25

25.2.3 Construction of an inscribed circle of a triangle

What is an inscribed circle of a triangle?

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.....

.....

Let's see if you have got this correctly. An inscribed circle of a triangle is a circle inside a triangle which touches the three sides of the triangle tangentially. See the diagram below

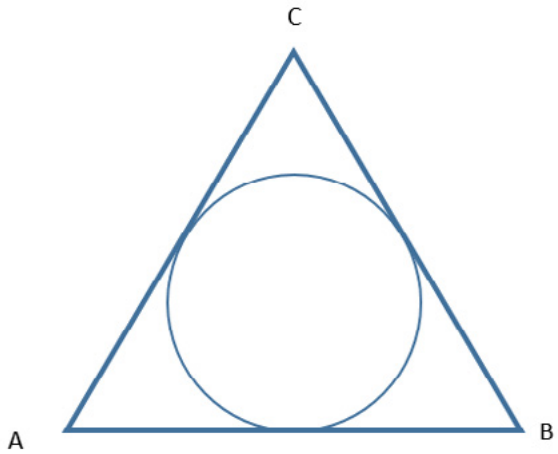


Fig 25.25

Now let us consider the following example on how we can construct a inscribed circle of a triangle

Worked Example [7]

Questions

(a) Construct ΔABC with AB=6cm, BC=6cm and AC=5cm

(b) construct an inscribed circle of ΔABC.

Solutions

Tip; You can bisect the 3 angles of the triangle or even 2 of the angles and they will intersect at a point, the point of intersection of the angle bisectors will be the centre of the circle

Step 1: Construct ΔABC (this must not be a problem for you at this point in time)

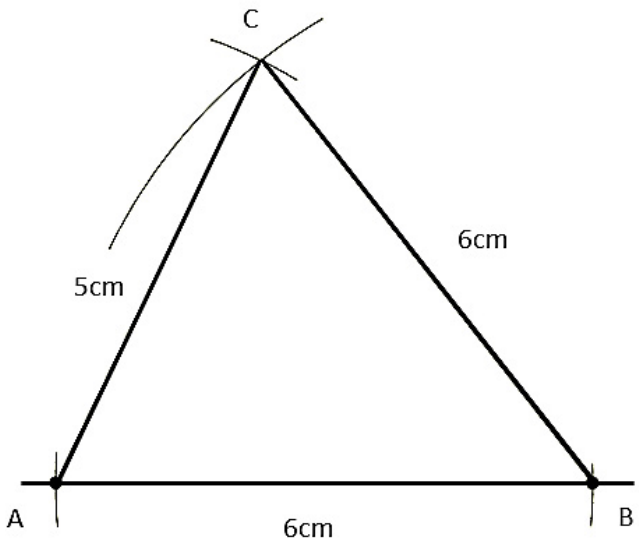


Fig 25.26

Step 2 - Bisect the three angle of the triangle

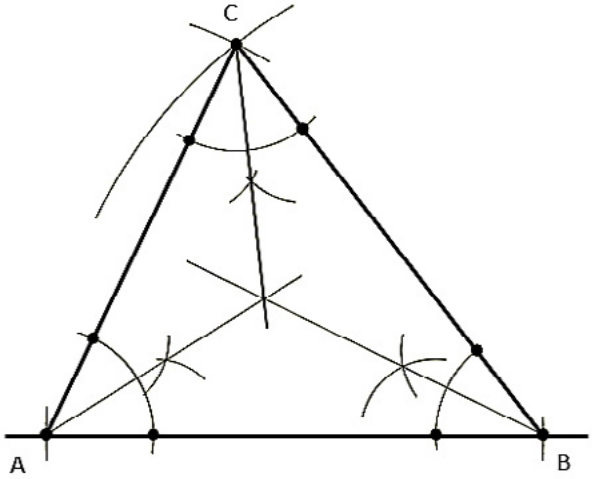


Fig 25.27

Step 3 - With the needle end at the point of intersection of the bisectors (the centre of the circle), open a radius from the centre to any edge of side of the triangle and draw a circle which touches the 3 sides

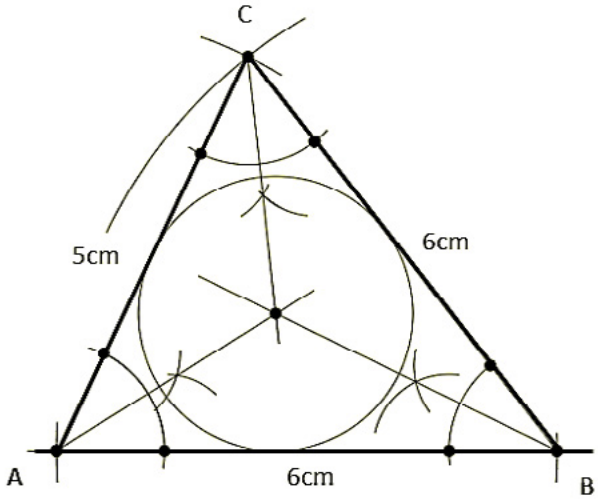


Fig 25.28

Practice

a) Construct a triangle ABC , with AB=6cm, ABC=120° and BC=8cm

b) Construct a inscribed circle of the above triangle

25.2.4 Construction of a circumscribed circle of a triangle

What is a circumscribed circle of a triangle?

.....

.....

.....

.....

Well this is a circle which touches all three vertices of a triangle

A circumscribed circle of a triangle is a circle drawn passing through the three corners of the triangle.

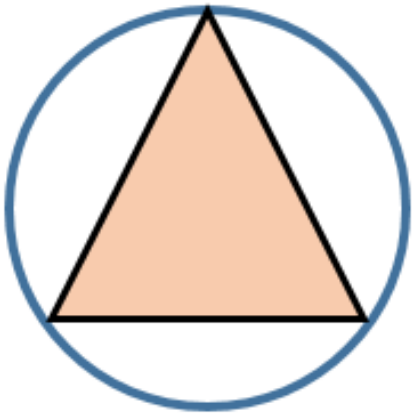


Fig 25.29

Now follow the steps in the example below on how the circumscribed circle can be constructed.

Worked Example [8]

Questions

(a) Construct ΔABC with AB=7cm, BC=6cm and AC=7cm

(b) Construct the circumscribed circle of the triangle above

Solutions

Step1 - Construct ΔABC (you are now familiar with constructing triangles

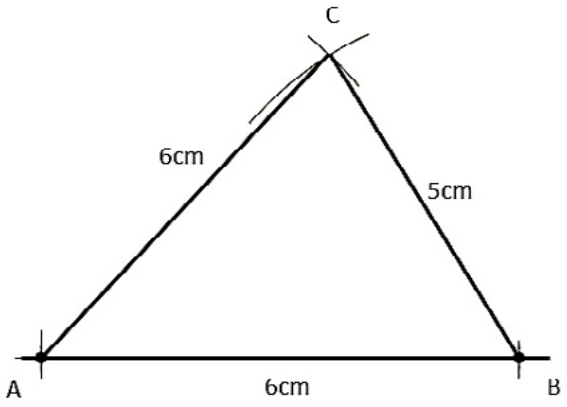


Fig 25.30

Step 2 - Bisect the three sides of the triangle or even two sides. They line bisectors will intersect at a point.

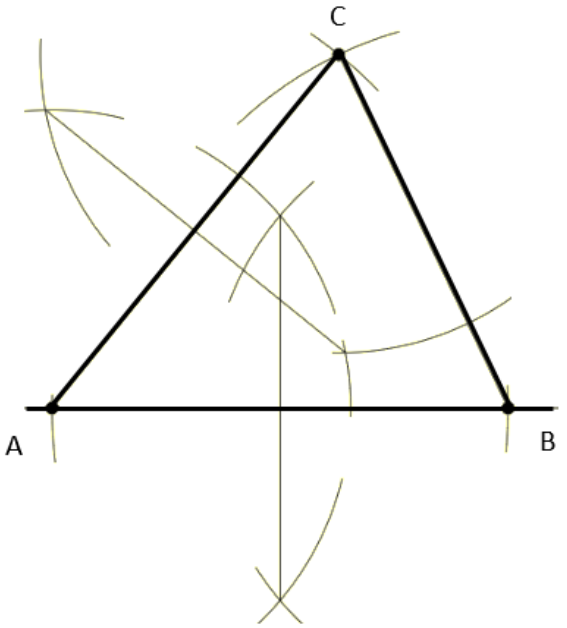


Fig 25.31

Step 3 - The point of intersection of the line bisector is the centre of the circle. Now with the needle end at the point of intersection, open a radius to one of the vertices say to A. (as the needle end is at the centre, the pencil end will be at point A)

Now draw a circle to pass through all the three corners.

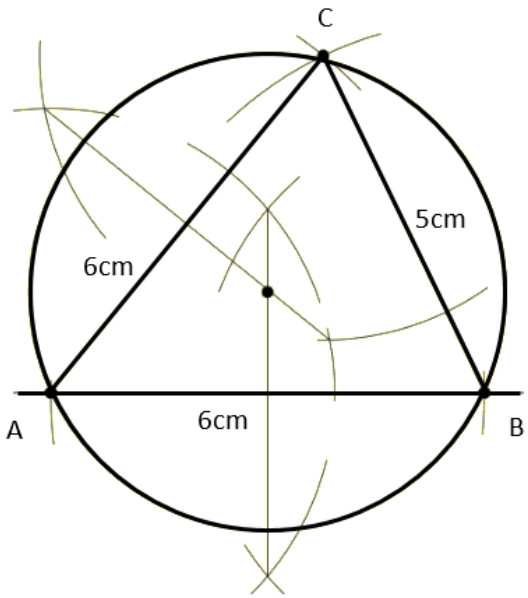


Fig 25.32

Practice

a) Construct a triangle ABC , with AB=6cm, ABC=120° and BC=8cm
b) Construct a circumscribed circle of the above triangle.

Activity (*25.3*) Circumscribed and inscribed circles

Questions

1) Construct triangle XYZ with XY=8cm, XYZ=90°and YZ=7cm
   On separate diagrams
   a) Construct the circumscribed circle of the triangle,
   b) Construct the inscribed circle of the triangle above.

2) Construct triangle ABC with AB=6.8cm, ABC=150° and BC=8cm. On separate diagrams
   a) Construct an inscribed circle of the triangle

b) Construct a circumscribed circle of the triangle above.

25.3 LOCUS

Definition

Locus can be defined as a set of points satisfying a given rule. It can also be defined as the path followed by a moving point. Loci is the plural for locus.

The following are the four laws of locus

25.3.1 The locus of points equidistant from a fixed point

What happens when a grazing goat is tied at a peg?

.....

.....

.....

Yes! It grazes within the radius of the given rope, in a circle. If the rope is 4cm long and the peg labelled O. It would be like this diagrammatically.

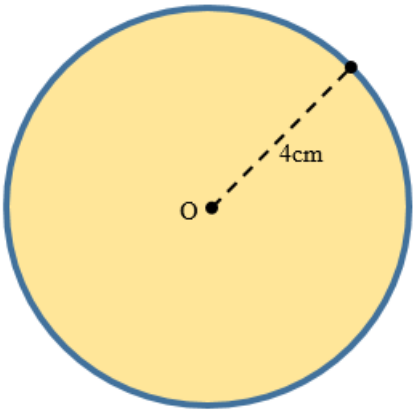


Fig 25.33

Therefore the locus of points equidistant from a fixed point is always a circle, x units from the fixed point.

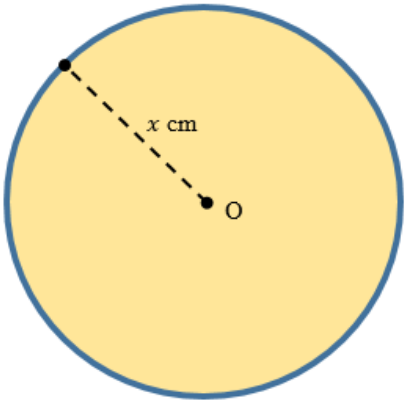


Fig 25.34

25.3.2 The locus of points equidistant from two fixed points

If A and B are two big trees and a straight road is to pass/cross in between. Try to sketch on the space below the position of the road.

If you are done sketching the path of the road. What did you observe?

.....

.....

.....

YES! You should observe that the road cross just in between A and B. Therefore it is the bisector of segment AB

Follow the example below as we go through constructing the bisector

Worked Example [9]

Questions

Given that line AB =8cm, construct the locus of points which are equidistance (at equal distance) from A and B

Solution

Step 1- Construct the line AB=8cm.

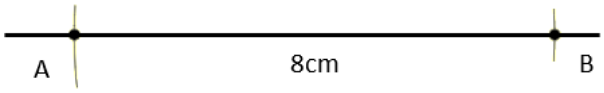


Fig 25.35

Step 2 - Bisect the line AB, thus making the line bisector.(which is the required line)

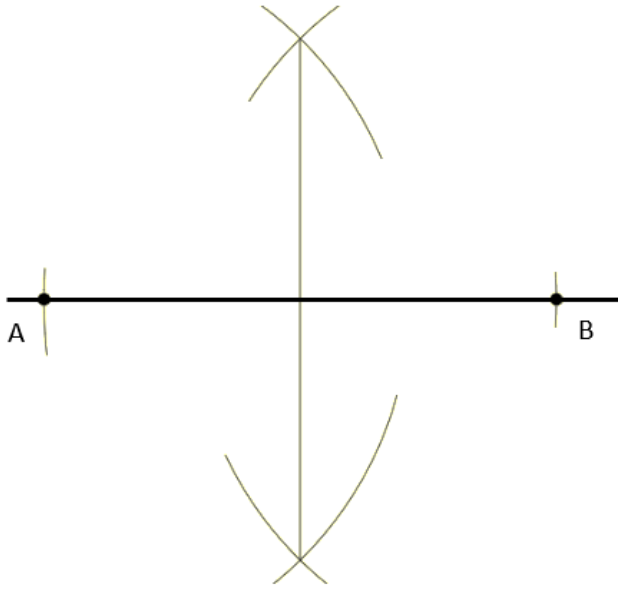


Fig 25.36

25.3.3 The locus of points equidistant from a straight line.

If a rope is to be laid on the ground and 10 small stones are to be placed 4cm from the rope where will there be? Show your answer on the diagram below

You should have shown something like this

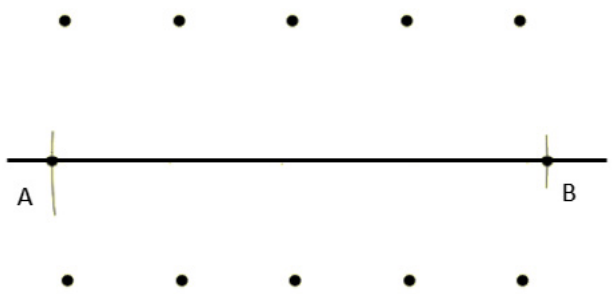


Fig 25.37

If these dots are connected they form a pair of parallel lines. Therefore the locus of points equidistant from a straight line forms a pair of parallel lines.

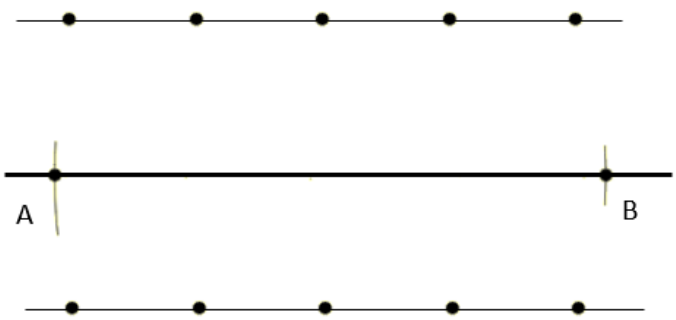


Fig 25.38

Let's look at the example below to see how these lines (parallel lines) are constructed

Worked Example [10]

Question

Construct the locus of points which are 4cm from line AB which is 8cm long

Solution

Step 1 - Construct a line AB=8cm (you are now familiar in construction of lines I hope)

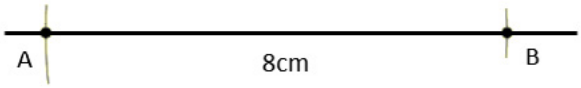


Fig 25.39

Step 2 - Open a radius of 4cm on your compass and with the needle end at A, make arc above and below A

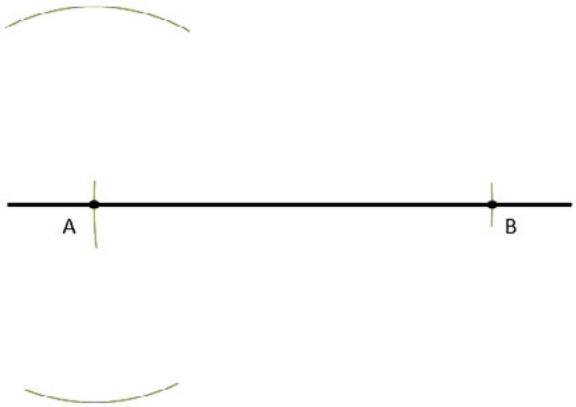


Fig 25.40

Step 3 - Without changing the radius, place the needle point at B and make an arc above and below point B

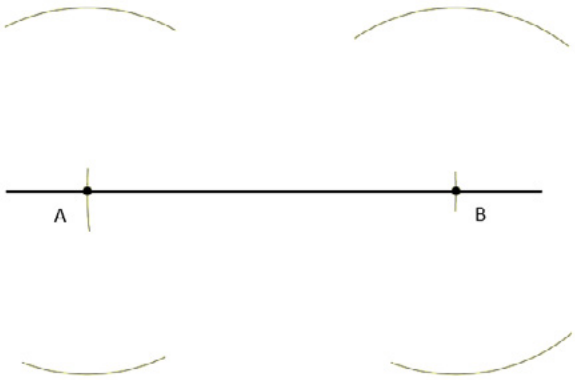


Fig 25.41

Step 4 - Draw straight lines which touches the arc tangentially as follows

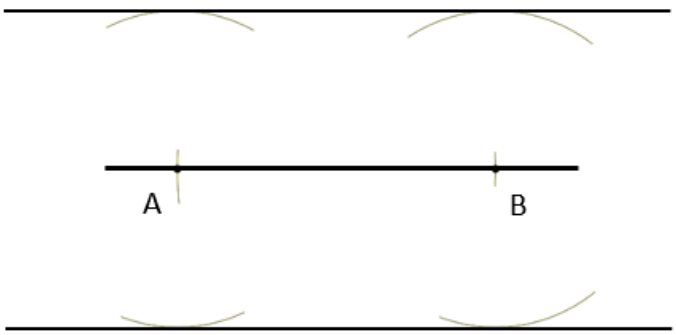


Fig 25.42

25.3.1 The locus of points equidistant from two intersecting straight lines

Suppose AB and AB are two intersecting straight lines as shown

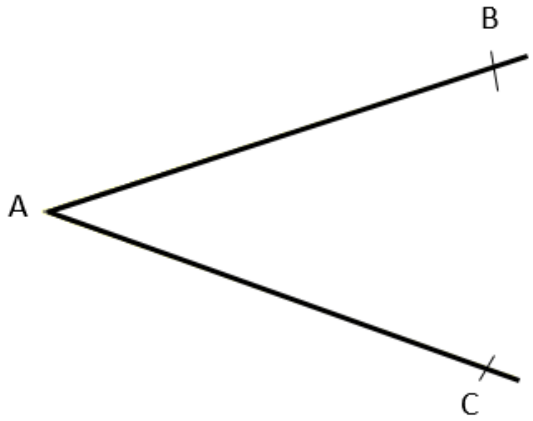


Fig 25.43

The locus of points equidistant (at equal distant) from AB and AC is an angle bisector of angle BAC.

Tip the two intersecting lines are AB and AC. Which letter is being repeated? Yes its A. Then just bisect the angle at the repeating letter as shown (hope by now you know how to bisect an angle)

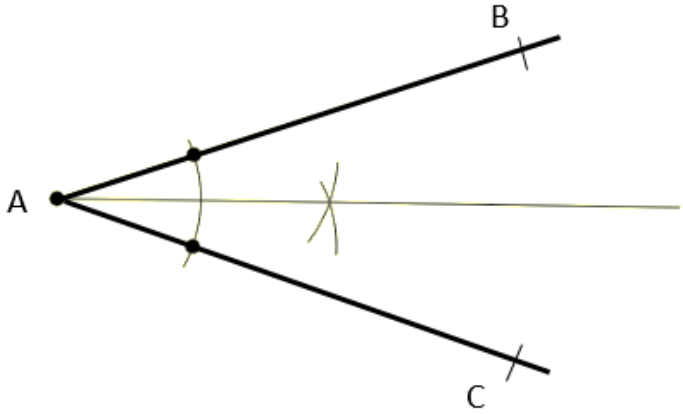


Fig 25.44

Activity (25.4) Loci

Questions

Describe the locus of points represented by the dotted line in each question.

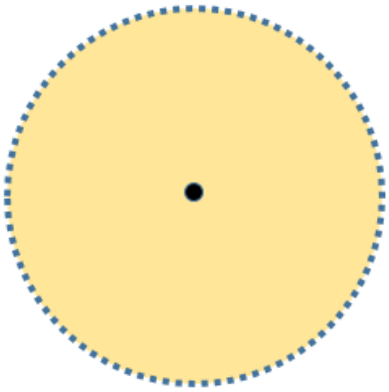


Fig 25.45

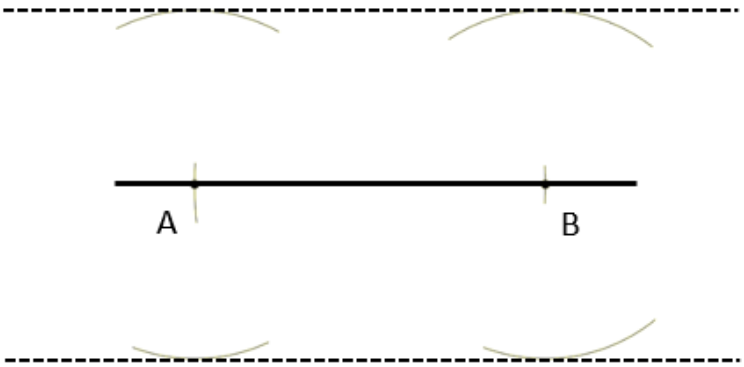


Fig 25.46

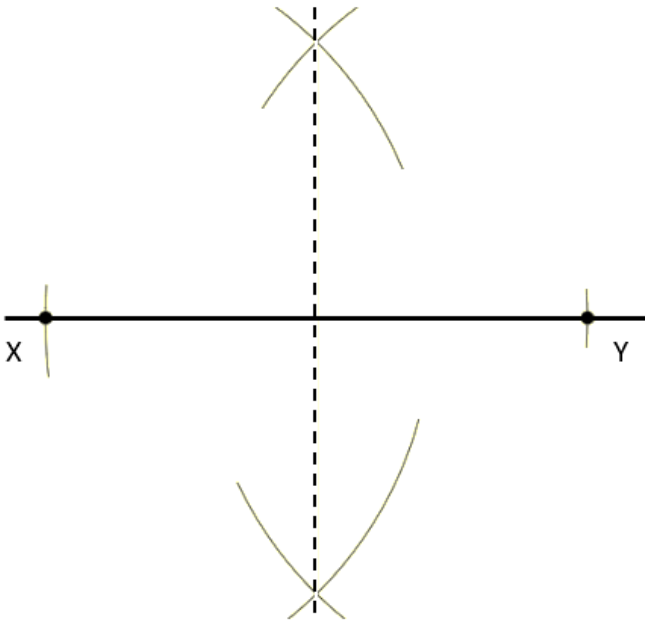


Fig 25.47

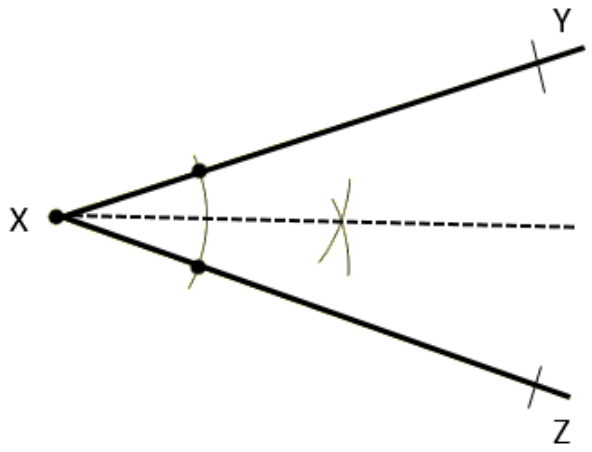


Fig 25.48

Application

Having done all this let Us try to answer questions which involve bearing. Follow the example below.

Worked Example [11]

Questions

Using ruler and compasses only for all constructions and show clearly all construction lines and arcs and on a single diagram.

In a new resettled area, facilities are described as follows;

From a dip tank (D), a grinding mill (G) is 5km due east, a borehole (B) is 3km away on a bearing of N30°W and a clinic (C) is due north. From the grinding mill the clinic is on a bearing N45°W.

a. Using a scale of 2cm to represent 1km, construct a diagram to show the positions of D, G, B and C.
   (8)
b. The chief's home is such that it is equidistant from the clinic, the borehole and the dip tank. Find, by construction, the possible position of the chief's home and label it H. (3)
c. Find the actual distance of the chief's home from the borehole.
   (2)

Solutions

Step 1 - Tip; first draw a sketch in the space below of the relative position of D, G, B and C

Step 2 - Draw an unmeasured straight line in the space below and label point D on your diagram, open a radius of 10cm and construct DC (we have converted km to cm using the given scale)

Step 3 - Make a 90°angle at point D, to show the position of C (due north of D) in the space below

Step 4 - In the space below, make a 90°angle at G and bisect it to make an angle of 45°(the 45°angle line from G, will intersect with the 90°angle line from D label the point of intersection C

Step 5 - In the space below, at D , make a 60°angle to the left of D to make the N30°W. Opening a radius of 6cm and make an arc and label C.

Step 6 - To find the position of the chief's home, we bisect the line BD, DC and BC, at the point of intersection of the line bisectors thus the chiefs home

Step 7 - To find the actual distance from the chief's home, we measure using a ruler and then we remember to change cm to km, using the given scale.

25.4 Summary

This unit has covered constructing quadrilaterals, constructing a perpendicular from a point to a line, constructing an inscribe circle of a triangle and the construction of a circumscribe circle of a triangle as well. On Loci, the unit covered the locus of points equidistant from a fixed point, the locus of points equidistant from two fixed points, the locus of points equidistant from a straight line and the locus of points equidistant from intersecting straight lines

25.5 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

25.6 Assessment Test

1. Use ruler and compasses only for all constructions and show clearly all the construction lines and arcs on a single diagram
   a. Construct a kite ABCD with AB=AD=6cm, BC=DC=9cm and BAD=60°. Join BD (6)
   b. Construct
      i. the locus of points equidistant from B and D
      ii. the locus of points equidistant from BD and DC. (3)
   c. The loci in (b) intersect at P. Using P as centre
      i. Draw a circle with BD as tangent,
      ii. Measure and write down the radius of the tangent (2)

2. Use ruler and compasses only for all construction ,
   A landmine infested area is in the form of a quadrilateral PQRS with PQ=14 km, QR=12km, PS=17km, PQR=90° and QPS=120°
   a. Using a scale of 1cm to represent 2km, construct quadrilateral PQRS (6)
   b. For safety reasons, resettled families are to be at least 6km from QR, Construct the locus of points 6km from QR. (1)
   c. Two landmines where located such that they were each equidistant from PS and SR
      And 10km from P.
      i. construct the locus of points equidistant from PS and SR'
      ii. construct the locus of points 10km from P (3)
   d. 1. Label M and N the two positions of the landmines. (1)
      2 find the actual distance between the landmines. (1)

UNIT 26 – STATISTICS

CONTENTS

26.1 INTRODUCTION

This unit seeks to equip you with the necessary skills required in statistical data collection, data analysis and data presentation. When you go through this unit you will be able to analyse and present measurements or observations done over a period of time using tables and graphs.

OBJECTIVES

After going through this unit you should be able to

- construct frequency tables
- draw histograms, frequency polygon and cumulative frequency curve
- find the median from cumulative frequency curve
- find the quartiles from cumulative frequency curve (Ogive)

KEY WORDS

Statistics - It is a form of mathematical analysis concerned with collecting, organizing, and interpreting data

Mean – it is the average of a set of numbers

Mode – It is the most frequently occurring number found in a set of numbers

Median – It is a simple measure of central tendency, that is, the middle value in a set of numbers arranged in rank order

Frequency – It is the number of times a data value occurs, that is, the number of occurrences of a particular item in a set of data

Discrete data – Refers to data that can only be represented by whole numbers for example number of people admitted at a hospital

Continuous data - This is information that does not take exact values for example age or weight of people in a family

Data – Refers to the facts and figures collected together for the purpose of analysis

Range – It is the difference between the lowest and highest values in a given set of data

Cumulative frequency – the total number of frequencies in a frequency distribution.

Quartiles – These are the values that divide data into quarters

TIME: 10 hours

STUDY SKILLS

Just like in the learning of probability, the key to understanding statistics is practically carrying out experiments. Knowledge of algebra also plays an important part in the learning of statistics. A graph book and a scientific calculator are very essential requirements when undertaking this unit.

26.2 STATISTICS – DATA COLLECTION

Statistics is a form of mathematical analysis concerned with collecting, organizing, and interpreting data. In statistics, data is represented by means of graphs. Now, let us look at how statistical information can be obtained.

Statistical information can be obtained through observation, measuring or even collecting the information from reliable sources. Let us assume that you are asked to obtain information on shoe sizes of members of your family and write them down and then asked to answer questions such as,

(i) What is the largest shoe size?

(ii) What shoe size is appearing most?

(iii) After arranging the shoe sizes in order, what is the middle shoe size?

Through answering the above questions you are analysing raw data, information which is not in a frequency table. The information on shoe size that you have collected is called raw data.

When analysing raw data you should be able to find the following measures:

(i) mean (ii) mode

(iii) median (iv) range

Let us now look into calculations related to data collection.

Worked Example [1]

Questions

Find the mean, mode and range of the following set of numbers

4, 9, 3, 8, 7, 11, 3, 5, 3, 8

Solutions

(a) Mean $= \frac{\text{sum}}{\text{n}}$

$= \frac{4+9+3+8+7+11+3+5+3+8}{10}$

$= \frac{61}{10} = 6{,}1$

(b) Mode = 3 (number appearing most)

(c) Range = largest value – smallest value

$= 11 - 3$

$= 8$

(d) Median =3, 3, 3, 4, 5,7, 8, 8, 9, 11

$= \frac{5+7}{2}$

$= 6$

It is after data has been collected that it can be classified. Let us look at how data is classified.

26.3 DATA CLASSIFICATION

When collecting statistical data, you collect it as raw data. Raw data is data that has not been placed in any group or category after collection. This data is therefore not sorted into categories and not classified. This data is usually found as a series of numbers. It is usually referred to as ungrouped data.
In order to easily interpret and analyse this data, this data should be classified. Data classification is the process of sorting and categorizing data into various types, forms or any other distinct class.

Tables and graphs are mostly used to classify raw data. When raw data has been classified then it becomes grouped data.

Worked Example [2]

Questions

The following raw data represents the number of cattle kept by individual families in Mutorahuku village.

0, 1, 2, 0, 0, 3, 0, 1, 6, 6, 5, 4, 1, 3, 3, 0, 1, 1, 2, 3, 3, 2, 3, 3, 4, 4, 6, 6, 6, 6.

In order for the data to be easily understood, it is placed in frequency tables as shows.

Number of cattle	Tally	Frequency
0	𝍸	5
1	𝍸	5
2	|||	3
3	𝍸 ||	7
4	|||	3
5	|	1
6	𝍸 |	6

Fig 26.1

Table 2 – Frequency table.

Number of cattle	0	1	2	3	4	5	6
Frequency	5	5	3	7	3	1	6

From the table above

(a) List the number of families in the village

(b) Calculate

(i) the mean number of cattle

(ii) the median number of cattle

(iii) the modal number of cattle

(iv) the range number of cattle

(c) Draw a bar chart to represent the information above.

Solutions

(a) 30 families

b) (i) Mean $= \frac{\text{Total number of cattle}}{\text{Total number of families}}$

$= \frac{0\times5+1\times5+2\times3+3\times7+4\times3+5\times1+6\times6}{5+5+3+7+3+1+6}$

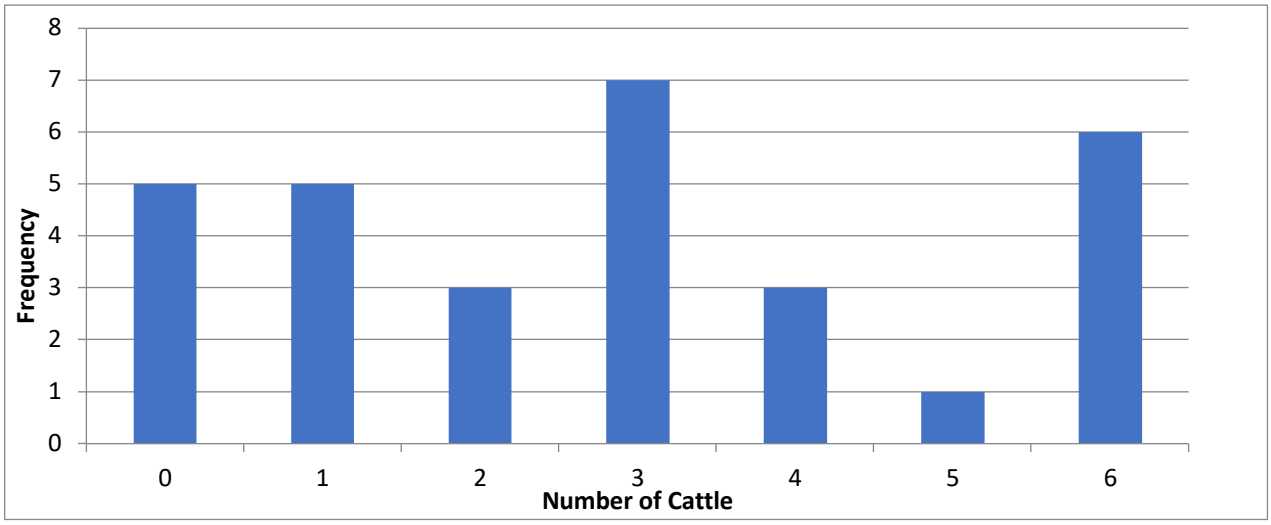
$= \frac{85}{30} = 2.8$

(ii) Median =3

(iii) Mode = 3

(iv) Range = 6 – 0 = 6

(c) Bar chart



Note that from the bar chart above, the bars are of equal width and have equal spaces in between.

Let us learn more about grouped and ungrouped data in the next example.

Worked Example [3]

Questions

Grouped data

The information below is ages of people kept at an old people's home:

71 82 83 76 103 79 83 104 77 83 86 102
96 102 75 96 75 82 85 76 79 84 88 95
99 75

The frequency table shows the raw data now grouped:

Number of people	Tally	Frequency
71 – 75	||||	4
76 – 80	~~||||~~	5
81 – 85	~~||||~~ | |	7
86 – 90	||	2
91 – 95	|	1
96 – 100	|||	3
101 – 105	||||	4

Fig 26.2

Table 3 – Frequency table

Age in years	71-75	76-80	81-85	86-90	91-95	96-100	101-105
Number of people	4	5	7	2	1	3	4

1) List the number of classes in the frequency table
2) Find the class centres of the above frequency table
3) Calculate the following
   a) Mean
   b) Modal class
   c) Mode

Solutions

1) 7 classes
2) The first class shows ages 71-75, its class centre is found as follows (75+71)÷2 = 73

Age in years	71-75	76-80	81-85	86-90	91-95	96-100	101-105
Number of people	5	6	8	3	1	3	4
Class centre	$= \frac{71+75}{2} = 73$	$= \frac{76+80}{2} = 78$	$= \frac{81+85}{2} = 83$	$= \frac{86+90}{2} = 88$	$= \frac{91+95}{2} = 93$	$= \frac{96+100}{2} = 98$	$= \frac{101+105}{2} = 103$

3) (a) To calculate the mean you multiply each class centre by its respective frequency:

Mean = sum of each class centre × respective class frequencies

$$= \frac{73\times5+78\times6+83\times8+88\times3+93\times1+98\times3+103\times4}{5+6+8+3+1+3+4}$$

= 85.3

b) Modal class = 81 – 85

c) Mode = $\frac{81+85}{2}$ = 83 years

After going through the above examples, attempt the following activity.

Activity (*26.1*) Mean. Mode, median and range

Questions

1) Find the mean, median, mode and range of the following set of numbers.
   (a) 3; 5; 5; 7; 7; 7; 9; 9;
   (b) 11; 10; 6; 5; 4; 12; 3; 6; 6

2) The table below represents ages of students in a class.

Age in years	12	13	14	15	16	17
Number of students	2	1	3	6	5	3

Calculate

(a) mean age
(b) median age
(c) modal age
(d) range

3) The set of numbers below are ages of patients admitted at children's ward at a general hospital.
   7; 4; 5; 7; 3; 2; 8; 7; 6; 5; 11; 9; 7; 3; 8; 7; 6; 5
   (a) Draw a frequency table to represent this information.
   (b) Calculate
      (i) Mean age
      (ii) Modal age
      (iii) Median age
      (iv) Range

4) The table below represents distances travelled by a group of people to the nearest health centre.

Distance in km	1-5	6-10	11-15	16-20	21-25
Number of people	2	11	14	10	3

(a) Calculate
   (i) mean distance travelled.
   (ii) Median distance.
   (iii) Modal class
   (iv) Mode

Answers

1 (a)

Mean = 6,5

Median = 7

Mode = 7

Range = 6

1 (b)

Mean = 7

Median = 6

Mode = 6

Range = 9

2

a) Mean = 15

b) Median = 15

c) Mode = 15

d) Range = 5

3 (a) Frequency table

Age	2	3	4	5	6	7	8	9	10	11
Frequency	1	2	1	3	2	5	2	1	0	1

3 (b)

(i). Mean = 6.11

(ii). Mode = 7

(iii). Median = 6.5

(iv). Range = 9

4 (a)

(i). Mean = 13,125

(ii). Median = 13

(iii). Modal class = 11–15

(iv). Mode = 13

26.4 HISTOGRAM AND FREQUENCY POLYGON

Usually grouped data is shown in either a histogram or frequency polygon. There are two types of histograms.

(i) histogram with equal class width

(ii) histogram with unequal class width

Worked Example [4]

Questions

The table shows the ages of people in a bus from Harare to Bulawayo.

Age (x) in years	10≤x<15	15≤x<20	20≤x<25	25≤x<30	30≤x<35	35≤x<40	40≤x<45
Number of people	3	5	6	4	3	4	2

Draw a

(a) histogram

(b) frequency polygon to represent the above information.

Solutions

$10 \leq x < 15$ has class width $15 - 10 = 5$. If you calculate the class widths of the other classes, you will find out that the class widths are the same.

Age (x) in years	10≤x<15	15≤x<20	20≤x<25	25≤x<30	30≤x<35	35≤x<40	40≤x<45
Number of people	3	5	6	4	3	4	2
Class width	15-10=5	20-15=5	25-29=5	30-25=5	35-30=5	40-35=5	45-40=5

The histogram is shown below. Note that the bars have no space between and that the bars have equal width.

26.4.1 Histogram with equal class width

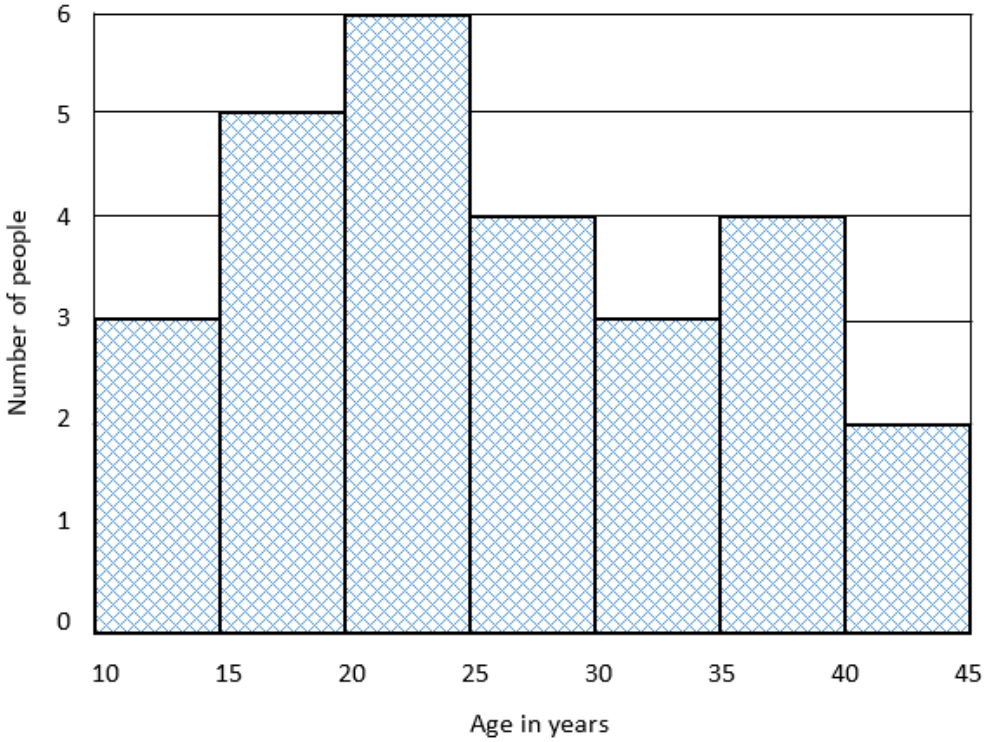


Fig 26.3

(c) For a frequency polygon, you mark the midpoint of each class, the point that corresponds with the frequency as shown, then join the points using a ruler.

26.4.2 Frequency polygon

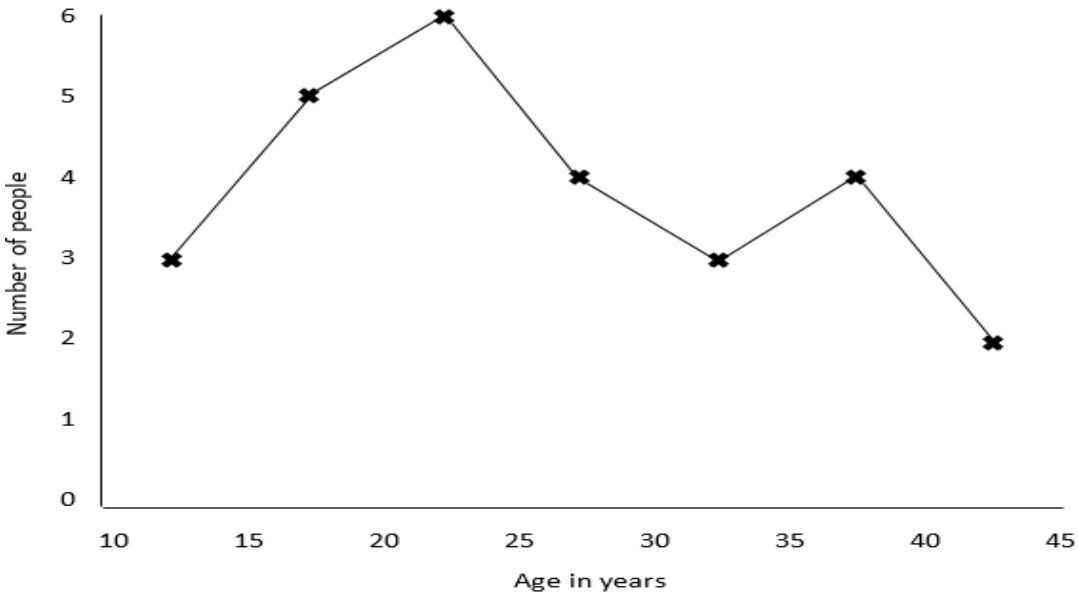


Fig 26.4

26.4.3 Histogram with unequal width

When the grouped data has unequal class width we use frequency density to draw the histogram.

Frequency density = $\frac{\text{Frequency}}{\text{class width}}$

Worked Example [4]

Question

The table below shows the masses of luggage belonging to travellers passing through the boarder.

Mass (m) in kg	0≤m<15	15≤m<20	20≤m<25	25≤m<30	30≤m<40	40≤m<50	50≤m<80
Frequency	15	10	30	10	30	40	30

Draw a histogram to show the above information.

Solution

The class widths are not equal so we use frequency density to draw the histogram. The frequency densities are calculated in the table below.

Mass(m)kg	0≤m<15	15≤m<20	20≤m<25	25≤m<30	30≤m<40	40≤m<50	50≤m<80
Frequency	15	10	30	10	30	40	30
Class width	15-0=15	20-15=5	25-20=5	30-25=5	40-30=10	50-40=10	80-50=30
Frequency density	$\frac{15}{15}=1$	$\frac{10}{5}=2$	$\frac{30}{5}=6$	$\frac{10}{5}=2$	$\frac{30}{10}=3$	$\frac{40}{10}=4$	$\frac{30}{30}=1$

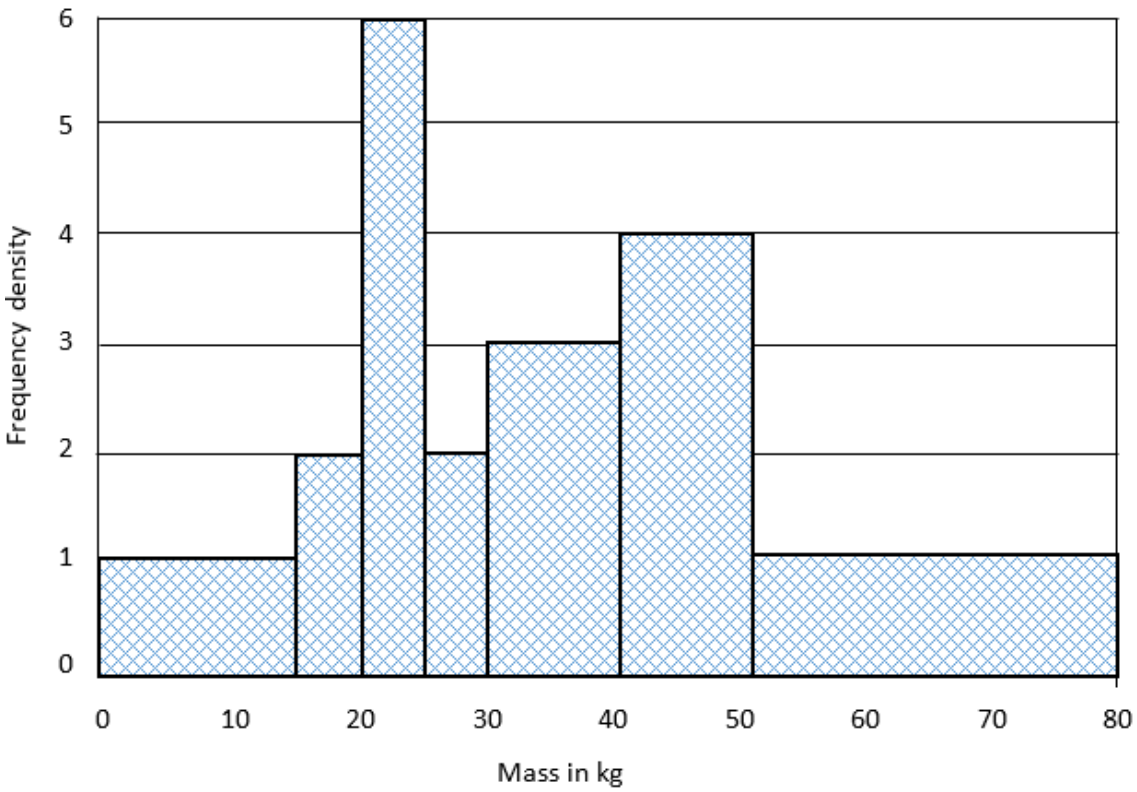


Fig 26.5

Now calculate area of each bar. Have you realised that the area of each bar gives you the frequency. Having done that now it's time for you to attempt the activity below.

Activity (*26.2*) Histograms and frequency polygons

Questions

Answer this question on a single sheet of graph paper

1. A group of 70 students were involved in a 50 km sponsored walk. The distances covered by the students are shown in the table below.

Distance x covered (km)	10< x ≤ 20	20< x ≤25	25< x ≤40	40< x ≤50
Number of students	30	12	w	13
Frequency density	3	v	1	1,3

(a) Find the value of

(i) v,

(ii) w.

(b) Using a scale of 2cm to represent 10km on the horizontal axis and 2 cm to represent 1 unit on the vertical axis, draw a histogram to represent the information on the table.

(c) State the modal class.

(d) A sponsor paid at the rate of $10000 per km. Calculate an estimate of the total amount paid to those who walked more than 25 km.

(e) Two students were chosen at random from the group. Calculate the probability that one walked at most 20 km and the other walked more than 20 km but less than or equal to 25 km. (ZIMSEC 2008)

2. The following distribution shows the masses of people admitted in a hospital:

Mass(kg)	40-45	45-60	60-70	70-75	75-85	85-100
Frequency	12	8	25	5	20	15
Frequency density	2,4	0,5	p	1	q	1

(a) Find the value of p and q

(b) Using a scale of 2cm to represent 10kg on the horizontal axis and 2cm to represent 1 unit on the vertical axis, draw a histogram to represent the information on the table.

(c) State the modal class.

(d) Calculate the mean of this distribution.

(e) When two people are chosen at random from this distribution, what is the probability that they have a mass between 40kg and 60 kg.

Solutions

1 (a) (i) $v = 2.4$

(ii) $w = 15$

(c) modal class = $10 < x \leq 20$

(e) $\frac{24}{161}$ or 0.149

2 (a) $p = 2.5$

(ii) $q = 2$

(c) modal class= $60 - 70$

(d) mean = 78.7

(e) $\frac{38}{5555}$ or 0,0685

26.5 CUMULATIVE CURVE

Cumulative frequency is obtained by adding the frequencies. The information will be used to draw a cumulative frequency curve or ogive. The cumulative frequency curve is very useful for calculating the median and the quartiles. This information will help you understand how the data is distributed when it is split into quarters.

Example:

Below are marks obtained by 40 students in a mathematics test.

9 8 4 2 8 10 9 7 5 0 1 8 9 7 7 10 2 3 4 4

6 2 5 9 10 8 2 5 3 1 4 9 8 2 4 5 4 3 8 4

Draw a cumulative frequency table for the above information.

Solution

Cumulative frequency table

Mark	Tally	Frequency	Cumulative Frequency
0	|	1	1
1	||	2	3
2	~~||||~~	5	8
3	|||	3	11
4	~~||||~~ ||	7	18
5	||||	4	22
6	|	1	23
7	|||	3	26
8	~~||||~~ |	6	32
9	~~||||~~	5	37
10	|||	3	40

Fig 26.6

The cumulative frequency tables can also be drawn as shown below.

Marks (x)	x≤0	x≤1	x≤2	x≤3	x≤4	x≤5	x≤6	x≤7	x≤8	x≤9	x≤10
Cumulative frequency	1	3	8	11	18	22	23	26	32	37	40

Worked Example [6]

Questions

The frequency table below represents amount of gold mined by 20 small scale miners on a particular day.

Mass in grams	1<x≤5	5<x≤10	10<x≤15	15<x≤20	20<x≤25	25<x≤30
Number of miners	1	3	7	5	3	1

(a) Find the values of a, b and c in the cumulative frequency table below

Mass in grams	≤5	≤10	≤15	≤20	≤25	≤30
Number of miners	1	4	a	b	c	20

(b) Using a scale of 2 cm to represent 5 grams on the horizontal axis and 2 cm represent 4 miners on the vertical axis, draw a cumulative frequency curve for the above information.

Solutions

(a) From above cumulative table you can see that only one miner mined less than 5 grams and 4 miners mined less than 10 grams. To find those who mined less than 15 grams = 1+3+7 =11, then a = 11. b=1+3+7+5=16 and lastly c=1+3+7+5+3=19.

(b) Using the cumulative frequencies you can now draw a cumulative frequency curve as shown below.

Cumulative frequency curve

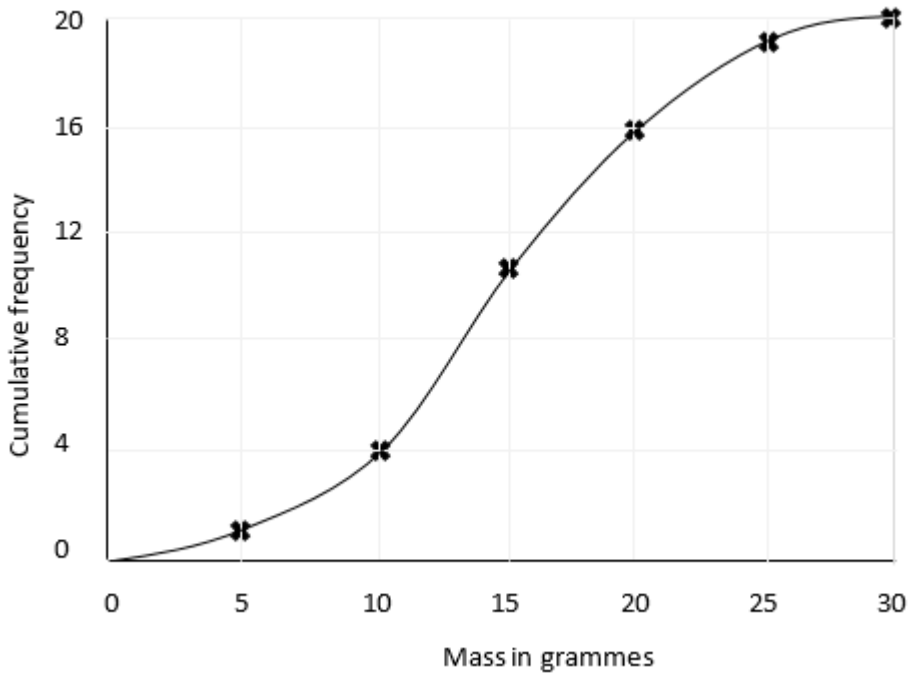


Fig 26.7

Hint:

When drawing a cumulative frequency curve,

- put the cumulative frequency on the vertical axis and the other quantities on the horizontal axis.
- Mark the cumulative frequencies that correspond with the upper boundary of each class.
- Join the points using a free hand so that you come up with a curve (ogive)

Worked Example [7]

Questions

Given below is a frequency distribution table showing the ages of 50 people in a certain village

Age in years	1-5	5-10	10-15	15-20	20-25	25-30
Number of people	6	4	15	12	8	5

(a) Draw a cumulative frequency table.

(b) Hence draw a cumulative frequency curve.

Solutions

(a) Cumulative frequency table

Age	Frequency	Cumulative frequency	Age limit
1-5	6	6	≤ 5
5-10	4	6+4=10	≤ 10
10-15	15	6+4+15=25	≤15
15-20	12	6+4+15+12=37	≤ 20
20-25	8	6+4+15+12+8=42	≤ 25
25-30	5	6+4+15+12+8+5=50	≤ 30

The cumulative frequency table above can be drawn simply as the table below.

Age (x) in years	x ≤5	x ≤ 10	x ≤ 15	x ≤ 20	x ≤ 25	x ≤ 30
Cumulative frequency	6	10	25	37	42	50

(b) The cumulative frequency above is then used to draw a cumulative frequency curve as shown below:

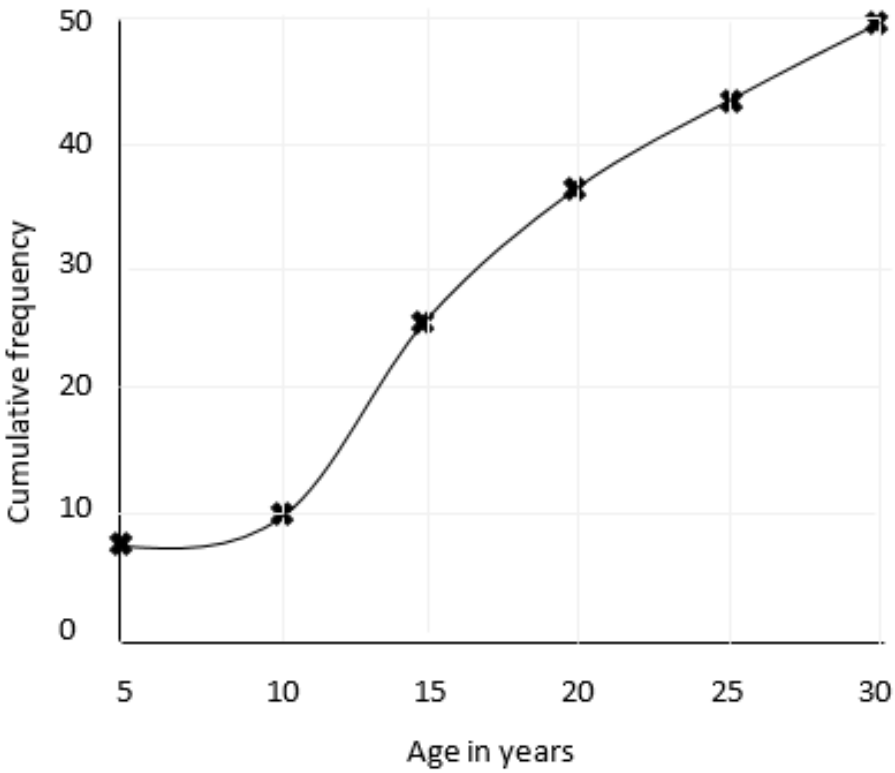


Fig 26.8

26.6 MEASURES OF DISPERSION

From a cumulative frequency curve, you should be able calculate the lower quartile (Q_1), the median (Q_2) and upper quartile (Q_3).

Quartiles are used to split the statistical data into quarters giving us the first quarter, second quarter and third quarter represented using Q_1, $Q_2$and Q_3 respectively.

Given the raw data representing ages of 15 students in a school bus

$$3, 3, 4, 5, 6, 6, 7, 7, 9, 10, 10, 10, 13, 14, 14$$

$5 \rightarrow Q_1$, $7 \rightarrow Q_2$, $10 \rightarrow Q_3$

Generally lower quartile (Q_1)

$= \frac{1}{4}(n+1)^{th}$ value

$= \frac{1}{4}(15+1)^{th}$

$= 4^{th}$ value

$=5$

Median (Q_2)

$= \frac{1}{2}(n+1)^{th}$ value

$= \frac{1}{2}(15+1)^{th}$

$= \frac{1}{2}(16)^{th}$

$= 8^{th}$ value

$=7$

Upper quartile (Q_3)

$= \frac{3}{4}(n+1)^{th}$ value

$= \frac{3(15+1)}{4}$

$= 12^{th}$value

$=10$

Inter quartile range

=Upper quartile- lower quartile

$=Q_3 - Q_1$

$=10-5$

$=5$

Semi inter-quartile range

$= \frac{\text{Upper quartile} - \text{lower quartile}}{2}$

$= \frac{10-5}{2}$

$= \frac{5}{2}$

$=2,5$

From the above analysis now try to calculate the quartiles using a cumulative frequency curve.

Worked Example [12]

Questions

During the first week of the month of August the following age groups were admitted at Rusape general hospital.

Age in years	41-45	46-50	51-55	56-60	61-65	66-70
Frequency	3	7	12	10	6	2

(a) Draw a

(i) cumulative frequency table and

(ii) hence draw a cumulative frequency curve to illustrate the above information

(b) Using the curve find

(i) median

(ii) lower quartile

(iii) upper quartile

(iv) inter quartile range

(v) Semi-inter quartile range

Solution

(a) (i) Cumulative frequency table

Age x in years	≤ 40	≤ 45	≤ 50	≤ 55	≤ 60	≤ 65	≤ 70
Cumulative frequency	0	3	10	22	32	38	40

(ii) Cumulative frequency curve

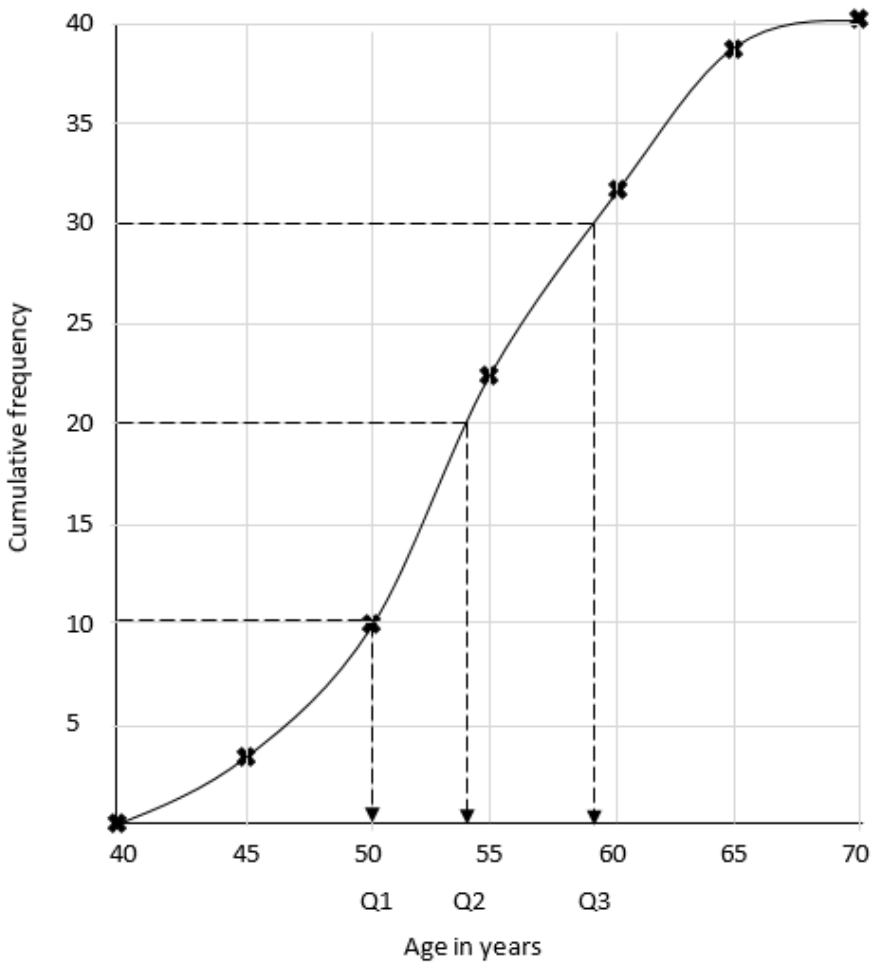


Fig 26.9

(b) Finding the

(i) median $(Q_2) = \frac{1}{2}(n+1)^{th} = \frac{40+1}{2} = 20{,}5^{th} = 54$years

(ii) Lower quartile $(Q_1) = \frac{1}{4}(n+1)^{th} = \frac{40+1}{4} = 10{,}25^{th} = 50$years

(iii) Upper quartile $(Q_3) = \frac{3}{4}(n+1)^{th} = \frac{3(40+1)}{4} = 30.75^{th} = 59$ years

(iv) Inter-quartile range $= Q_3 - Q_1 = 59 - 50 = 9$ years

(v) Semi-inter quartile range $= (Q_3 - Q_1) \div 2 = \frac{59-50}{2} = 4.5$ years

For the above answers allowance of ±0,2 is acceptable.

Now it's time for you to go through the self-assessment activity below

26.7 Summary

From what you have done in this unit you are now able to draw both the frequency distribution table and cumulative frequency table. You are now also able to represent statistical information on a histogram, frequency polygon and cumulative frequency curve. With the skills you have acquired in this unit you can analyse statistical information given as raw data, frequency table or in graphical form. Now move on to the next unit.

26.8 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

26.9 Assessment Test

Questions

Answer the whole of this question on a single sheet of graph paper

Weight Class(Kg)	Frequency	Cumulative frequency
26-30	4	4
31-35	6	10
36-40	12	22
41-45	10	32
46-50	7	p
51-55	1	40

The incomplete table shows the grouped frequency of the weights to the nearest kilogram, of a group of 40 people.

a) Write down (i) the value of p,
(ii) the modal class

b) Using a horizontal scale of 2cm to represent 5 kilograms and a vertical scale of 2cm to represent 5 people, draw a cumulative frequency curve (ogive) to show this information.

c) Using your cumulative frequency curve find an estimate of the median weight.

d) One person is picked at random. Find the probability that this person is in the 36 to 40 kg range.

e) Given also that 30% of the people are female, find the probability that two people chosen at random, from the group, are both female. (ZIMSEC 2009)

Answers:

(a) (i) p = 39 (ii) modal class = 36-40 (c) 39 (d) $\frac{3}{10}$ (e) $\frac{11}{130}$

UNIT 27 - TRANSFORMATION 2

CONTENTS

27.1 INTRODUCTION

This unit seeks to address the myths and challenges associated with use of matrices in solving transformations. The unit will cover in detail the following transformations; reflection, rotation and enlargement. It is also going to look into combined transformations.

OBJECTIVES

After going through this unit, you should be able to

- describe the types transformations
- draw the images of shapes and figures under given transformations

KEY TERMS

Isometric: – the shape produced after transformation maintains the same dimensions as the original shape, that is, shape remains congruent to the original shape after transformation.

Non-Isometric: – the shape produced after transformation does not maintain the same dimensions as the original shape, that is, it changes shape after transformation.

TIME:-You are expected not to spend more than 8 hours on this unit.

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems in Transformation as possible for you to grasp all the concepts in this topic.

27.2 REFLECTION IN ANY LINE AND USE OF MATRICES

27.2.1 Reflection in any line (Geometrical solution)

In this concept we need to focus much on finding the mirror line or the reflection line by drawing. Let us consider an example where we see how a reflection line is obtained.

Worked Example [1]

Questions

Using a scale of 2cm to 1 unit draw x-axis and y-axis numbered from 0 to 9 mark the following points X(4;7), Y(6;7), Z(7;5) and the images $X_1(2;5)$, $Y_1(2;3)$, $Z_1(4;2)$ under a reflection.

(a) Draw the reflection line

(b) Find the equation of reflection.

Solutions

Tips

- Plot all points as required
- Join corresponding points and identify their centres by geometrical construction of perpendicular bisectors or by measurement

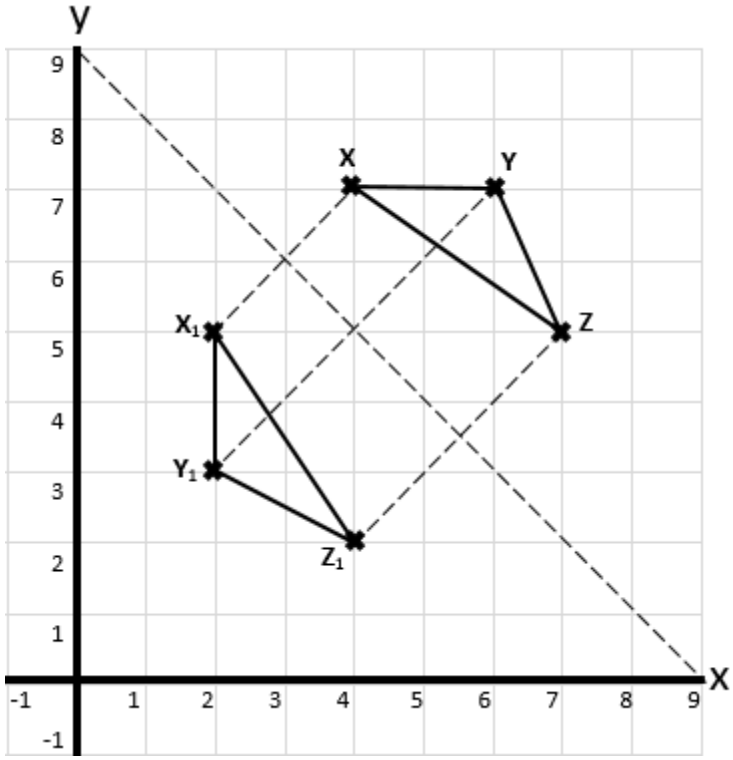


Figure 27.1

(b)

TIPS

- Find the gradient of the perpendicular bisector and the y – intercept ($y = mx + c$) or use any other means to find the equation of the line.

  Consider and two points from the line, say (3;6) and (7;2)

  $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{3-7} = \frac{4}{-4} = -1$$

  $c = 9$

  By substitution in $y = mx + c$

  Therefore, the reflection line is $y = -x + 9$

27.2.2 Use of Matrices in Reflections

We may start by deriving the matrices which are obtained by reflecting along the x-axis, y-axis, $y = x$ and $y = -x$ as these are really vital at your Ordinary Level course

Worked Example [2]

Questions

The figure below shows objects reflected $\mathbf{y} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$. Find the reflection matrices used to transform

a) ΔA to ΔA_1

b) ΔA to ΔA_2

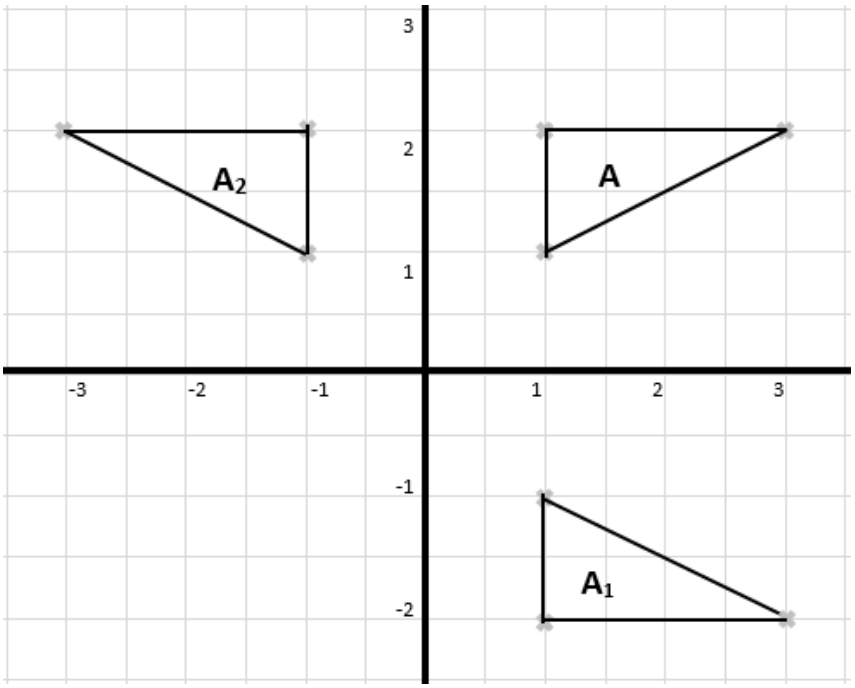


Figure 27.2

Solutions

a) Let us consider ΔA with points (1;1), (1;2) and (3;2) and the image ΔA_1 with (1;-1), (1;-2) and (3;-2).

There must be a 2×2 matrix of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which was used to reflect points from ΔA to ΔA_1 along $\mathbf{y} = \mathbf{0}$ and this can be derived as follows for all the points

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Four linear equations can be created from the matrices and be solved as follows

$a + b = 1 \ldots\ldots (1)$
$c + d = -1 \ldots\ldots (2)$
$a + 2b = 1 \ldots\ldots (3)$
$c + 2d = -2 \ldots\ldots (4)$

By solving equation 1 and 3 simultaneously as follows

$$\begin{array}{l} (a + 2b = 1) \\ \underline{-(a + b = 1)} \\ \qquad\quad b = 0 \text{ and} \\ a + 0 = 1 \\ a = 1 \end{array}$$

and by solving equations 2 and 4 simultaneously

$$\begin{array}{l} (c + d = -1) \\ \underline{-(c + 2d = -2)} \\ \qquad\quad d = -1 \text{ and} \\ c + (-1) = -1 \\ c = 0 \end{array}$$

Therefore, the reflection matrix along y = 0 is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Note: The same can be done when an object is reflected along the line $x = 0$

b) Let us also consider ΔA with points (1;1), (1;2) and (3;2) and the image ΔA_2 with (-1;1), (-1;2) and (-3;2), there must be a 2×2 matrix of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which was used to reflect points from ΔA to ΔA_2 along $\mathbf{x} = \mathbf{0}$ and this can be derived as follows for all the points

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

Four linear equations can be created from the matrices and be solved as follows

$a + b = -1 \ldots\ldots (1)$

$c + d = 1 \ldots\ldots\ldots (2)$

$a + 2b = -1 \ldots\ldots\ldots (3)$

$c + 2d = 2 \ldots\ldots\ldots (4)$

By solving equation 1 and 3 simultaneously,

$$\begin{array}{l} (a + 2b = -1) \\ \underline{-(a + b = -1)} \end{array}$$

$a = -1$ and $b = 0$

and by solving equations 2 and 4 simultaneously

$$\begin{array}{l} (c + d = 1) \\ \underline{-(c + 2d = 2)} \end{array}$$

$c = 0$ and $d = 1$

Therefore, the reflection matrix along $\mathbf{x} = \mathbf{0}$ is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Worked Example [3]

Questions

The figure below shows objects reflected along $\mathbf{y} = -\mathbf{x}$ and $\mathbf{y} = \mathbf{x}$. Find the reflection matrices used to transform

a) ΔABC to $\Delta A_1B_1C_1$

b) ΔXYZ to $\Delta X_1Y_1Z_1$

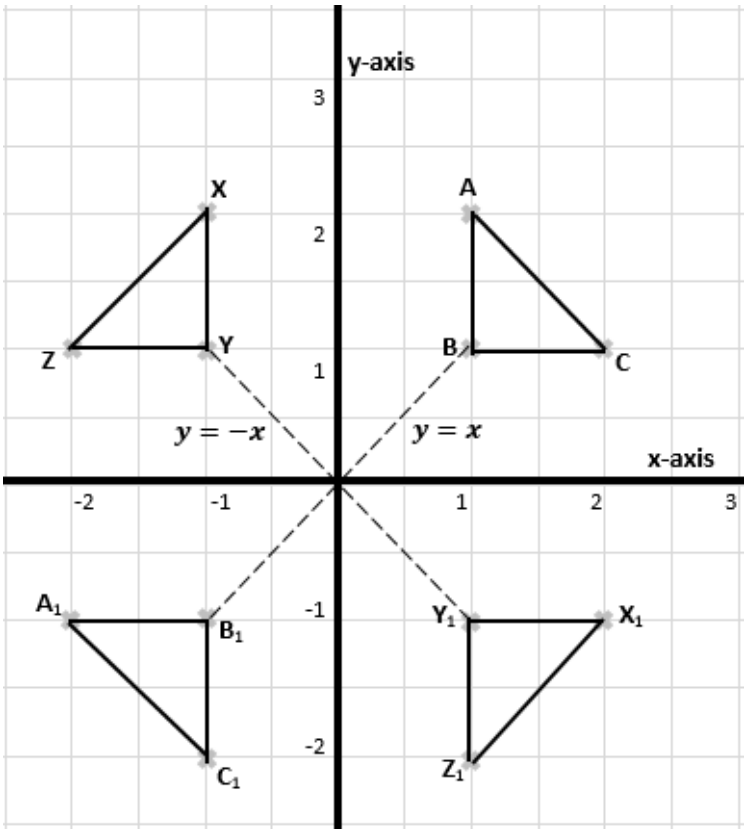


Fig 27.3

Solutions

a) Let us consider ΔABC in figure 25.3 with points (1;2), (1;1) and (2;1) and the image $\Delta A_1B_1C_1$ with (-2;-1), (-1;-1) and (-1;-2) respectively. As you can see that is reflection along the line $\mathbf{y} = -\mathbf{x}$ and the matrix can be derived as follows.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

Four linear equations can be created from the matrices and be solved as follows

$a + 2b = -2 \ldots\ldots (1)$
$c + 2d = -1 \ldots\ldots (2)$
$a + b = -1 \ldots\ldots (3)$
$c + d = -1 \ldots\ldots (4)$

By solving equation 1 and 3 simultaneously,

$$\begin{array}{l} (a + 2b = -2) \\ \underline{-(a + \ \ b = -1)} \\ \qquad\quad b = -1 \ \text{ and} \end{array}$$

$$a + 2(-1) = -2$$
$$a = 0$$

and by solving equations 2 and 4 simultaneously

$$\begin{array}{l} (c + 2d = -1) \\ \underline{-(c + \ \ d = -1)} \\ \qquad\quad d = 0 \text{ and} \end{array}$$

$$c + 0 = -1$$
$$c = -1$$

Therefore, the reflection matrix along $\mathbf{y} = -\mathbf{x}$ is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

b) Let us also consider ΔXYZ in figure 25.3 with points (-1;2), (-1;1) and (-2;1) and the image $\Delta X_1Y_1Z_1$ with (2;-1), (1;-1) and (1;-2) respectively. As you can see that is reflection along the line $\mathbf{y} = \mathbf{x}$ and the matrix can be derived as follows.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Four linear equations can be created from the matrices and be solved as follows

$-a + 2b = 2 \ldots\ldots (1)$
$-c + 2d = -1 \ldots\ldots (2)$
$-a + b = 1 \ldots\ldots\ldots (3)$
$-c + d = -1 \ldots\ldots (4)$

By solving equation 1 and 3 simultaneously,

$(-a + 2b = 2)$
$\underline{-(-a + b = 1)}$
$b = 1$ and
$-a + 2(1) = 2$
$a = 0$

and by solving equations 2 and 4 simultaneously
$(-c + 2d = -1)$
$-(-c + d = -1)$
$d = 0$ and
$-c + 2(0) = -1$
$c = 1$

Therefore, the reflection matrix along $\mathbf{y} = \mathbf{x}$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Table 25.1 - Summarising table of reflection matrices

Transformation	Mirror line(equation)	Matrix
Reflection	$x - \text{axis } (y = 0)$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection	$y - \text{axis } (x = 0)$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection	$y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Reflection	$y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Worked Example [4]

Questions

Given that triangle ABC, has point $A(8; 2), B(8; 6), C(6; 6)$. Reflect each point of the triangle along the line $y = 0, x = 0, y = x$ and $y = -x$ using the correct matrices for each equation. Show clearly each image point after transformation

Solutions

Remember ($\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for a reflection along $y = 0$**)**

$$\begin{array}{cccc} & A & B & C \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 8 & 8 & 6 \\ 2 & 6 & 6 \end{pmatrix} & = & \end{array} \begin{array}{ccc} A_1 & B_1 & C_1 \\ \begin{pmatrix} 8 & 8 & 6 \\ -2 & -6 & -6 \end{pmatrix} \end{array}$$

Therefore the new image points are $A_1(8;-2)$, $B_1(8;-6)$ and $C_1(6;-6)$

Remember ($\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ for a reflection along $x = 0$**)**

$$\begin{array}{cccc} & A & B & C \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 8 & 8 & 6 \\ 2 & 6 & 6 \end{pmatrix} & = & \end{array} \begin{array}{ccc} A_1 & B_1 & C_1 \\ \begin{pmatrix} -8 & -8 & -6 \\ 2 & 6 & 6 \end{pmatrix} \end{array}$$

Therefore the new image points are $A_1(-8;2)$, $B_1(-8;6)$ and $C_1(-6;6)$

Remember ($\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for a reflection along $y = x$**)**

$$\begin{array}{cccc} & A & B & C \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 8 & 8 & 6 \\ 2 & 6 & 6 \end{pmatrix} & = & \end{array} \begin{array}{ccc} A_3 & B_3 & C_3 \\ \begin{pmatrix} 2 & 6 & 6 \\ 8 & 8 & 6 \end{pmatrix} \end{array}$$

Therefore the new image points are A_1 (2;8), B_1 (6;8) and C_1 (6;6)

Remember ($\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ for a reflection along $y = -x$**)**

$$\begin{array}{cccc} & A & B & C \\ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 8 & 8 & 6 \\ 2 & 6 & 6 \end{pmatrix} & = & \end{array} \begin{array}{ccc} A_1 & B_1 & C_1 \\ \begin{pmatrix} -2 & -6 & -6 \\ -8 & -8 & -6 \end{pmatrix} \end{array}$$

Therefore the new image points are $A_1(-2;-8)$, $B_1(-6;-8)$ and $C_1(-6;-6)$

Activity (*27.1*) Reflection

Questions

1. Triangle XYZ has vertices X(0;0), Y(2;-2) and Z(2;2). Use the appropriate matrix to reflect the triangle along the x-axis.
2. A rhombus has vertices (2;2), (1;-1), (-2;-2) and (-1;1). Using the appropriate matrices find the coordinates of the its vertices when it is reflected along the
   (a) Line y = x
   (b) Line y = - x
   (c) x – axis
   (d) y – axis

3. Answer the whole of this question on a sheet of graph paper
   (a) Draw a Cartesian plane with $-9 \leq x \leq 9$ and $-9 \leq y \leq 9$
   (b) Draw and label triangle ABC with A(5;6), B(1;7) and C(1;5)
   (c) Draw the lines y = 2 and y = x
   (d) Draw the image of triangle ABC after reflection in
      (i) y = 2
      (ii) y = x
   (e) Write the matrix of the image in each case

Solutions

1. X_1(0;0), Y_1(2;2) and Z_1(2;-2).

2. (a) (2;2), (-1;1), (-2;-2) and (1;-1)
(b) (-2;-2), (1;-1), (2;2) and (-1;1)
(c) (2;-2), (1;1), (-2;2) and (-1;-1)
(d) (-2;2), (-1;-1), (2;-2) and (1;1)

27.3 ROTATION BY DRAWING AND USE OF MATRICES

This concept has been covered in Unit 23. In that unit we have explained that rotations as isometric, that is, all shapes or objects rotated will always maintain their equivalent sizes of the original shape. Again we have stated that all objects are rotated either clockwise or anticlockwise about a fixed point (x;y) called the centre of rotation.

Now we need to take rotation to a higher level where we are going to make use of drawings and matrices in calculations of centres of rotation and image points

27.3.1 Rotation by drawing (Geometrical construction)

In most cases learners face challenges in picking the centre and angle of rotation of an object. Let us take our geometrical/ or mathematical instruments and get down to business.

Tips: The following steps should be followed when finding the centre angle of rotation

- Use a ruler to joint corresponding rotated points (pre-image and image points), say X against X^1 and Y against Y^1.
- Construct the perpendicular bisectors of the joined points
- The perpendicular bisectors should meet at the centre of rotation
- Use a protractor to measure the angle of rotation, set your protractor properly on the centre of rotation and count degrees from zero(0)

Worked Example [5]

Questions

Given that triangle ABC with vertices A(4;8), B(4;4) and C(6;6) is rotated onto the image triangle $A_1B_1C_1$ with vertices A_1(-6;-2), B_1(-2;-2) and C_1(-4;0).

(a) Use a suitable scale of your choice to draw the two triangles.

(b) Find the centre and angle of rotation by construction.

Solutions

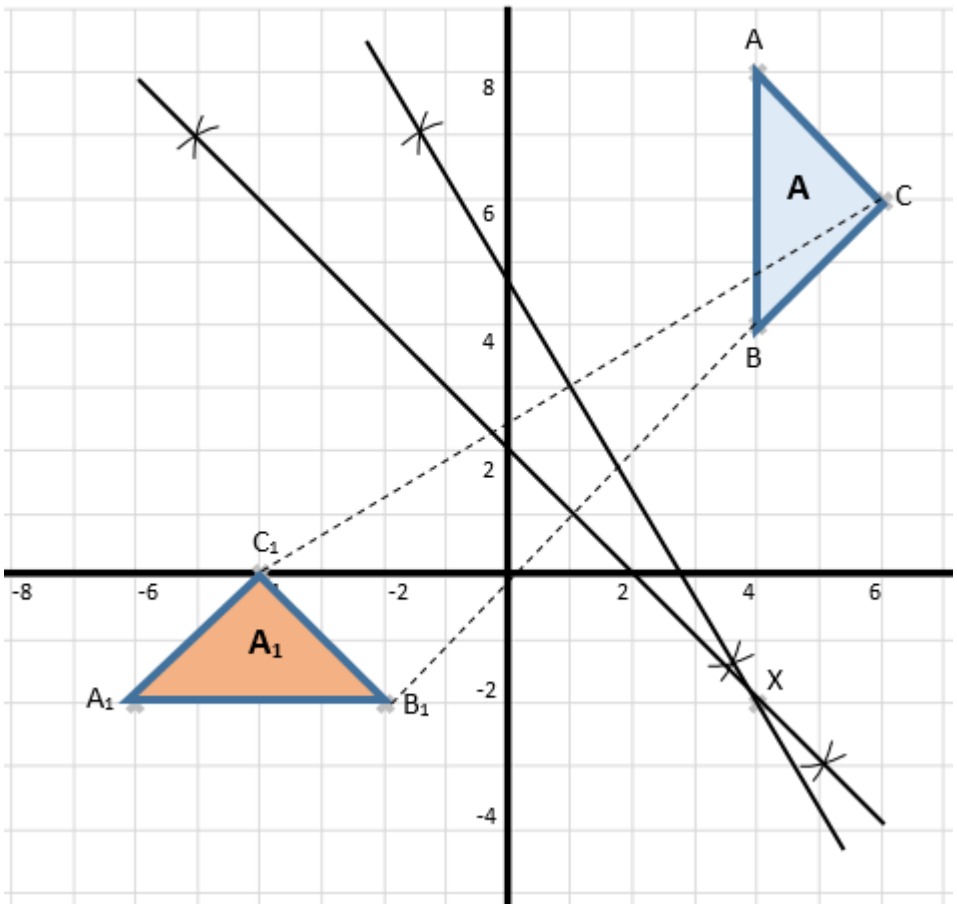


Fig 27.4

The centre X of rotation is (4;-2) and the angle of rotation is 90° anticlockwise or 270° clockwise.

27.3.2 Matrices in Rotation transformation

Again this is a major concept in transformation where we need to derive all matrices of rotation transformation.

Worked Example [6]

Questions

Figure 27.5 below shows rotation of four clockwise rotations of 90°, 180° and 270° and the following are some matrices derived for you

Let us consider ΔABC in figure 27.5 with points (1;2), (1;1) and (2;1) and the image $\Delta A_1B_1C_1$ with (2;-1), (1;-1) and (1;-2) respectively. As you can see that is a rotation of 90° about the origin and the matrix can be derived as follows.

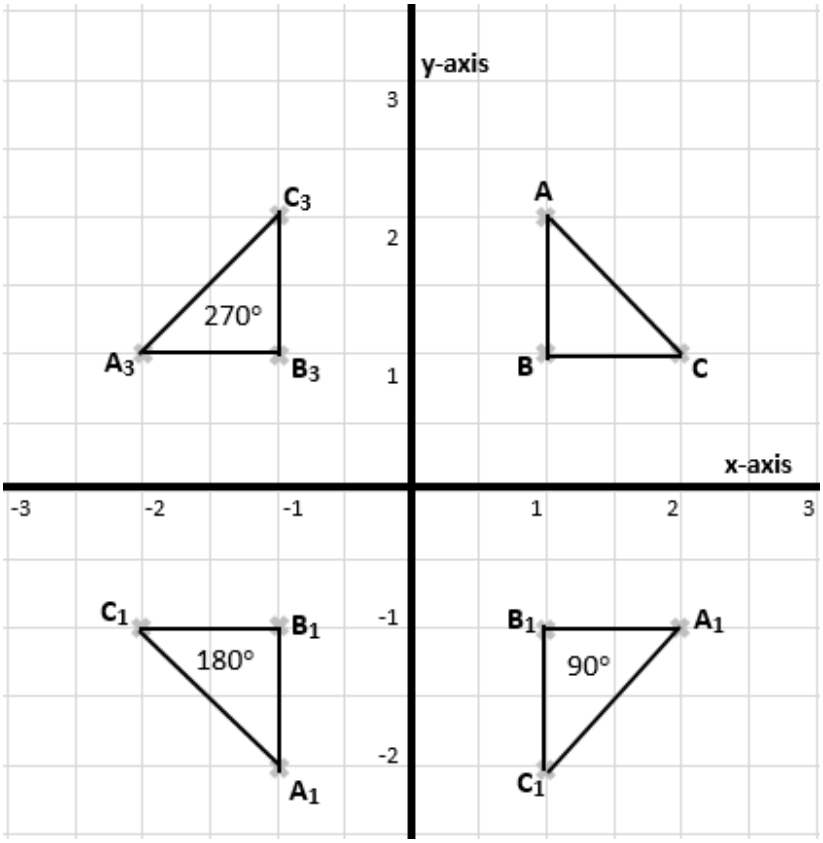


Fig. 27.5

Solutions

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Four linear equations can be created from the matrices and be solved as follows

$a + 2b = 2 \ldots . (1)$

$c + 2d = -1 \ldots . . (2)$

$a + b = 1 \ldots (3)$

$c + d = -1 \ldots . (4)$

By solving equation 1 and 3 simultaneously,

$(a + 2b = 2)$

$\underline{-(a + b = 1)}$

$b = 1$ and

$a + 2(1) = 2$

$a = 0$

and by solving equations 2 and 4 simultaneously

$(c + 2d = -1)$

$\underline{-(c + d = -1)}$

d = 0 and

c + 2(0) = - 1

c = - 1

Therefore, a **clockwise** rotation matrix of **90°** about the origin is R $=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, generally that's how all matrices can be obtained. However, we can also derive the other matrices using the method below.

A **clockwise** Rotation matrix of **180°** about the origin is

RR$=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

A **clockwise** Rotation matrix of **270°** about the origin is

RRR$=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

A **clockwise** Rotation matrix of **360°** about the origin is

RRRR$=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Note: Rotation matrix of **360°** about the origin gives the initial object.

Given that an **anticlockwise** Rotation matrix of **90°** about the origin is R $=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

An **anticlockwise** Rotation matrix of **180°** about the origin is

RR$=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

An **anticlockwise** Rotation matrix of **270°** about the origin is

RRR$=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Table 27.2

Transformation	Description	Matrix
Rotation	90° clockwise about the origin (0;0)	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Rotation	180° about the origin (0;0)	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
Rotation	270° clockwise about the origin (0;0)	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Rotation	90° anticlockwise about the origin (0;0)	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Rotation	270° anticlockwise about the origin (0;0)	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Note:

- 180° clockwise is equal to 180° anticlockwise
- 90° clockwise is equal to 270° anticlockwise
- 270° clockwise is equal to 90° anticlockwise

Worked Example [7]

Questions

Given that triangle PQR, has point $P(2;2), Q(3;6), R(-4;7)$. Rotate the triangle using the given matrices, state the new points after rotation and the angles of rotation

(a) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Solutions

(a) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} P & Q & R \\ 2 & 3 & -4 \\ 2 & 6 & 7 \end{pmatrix}=\begin{pmatrix} P_1 & Q_1 & R_1 \\ 2 & 6 & 7 \\ -2 & -3 & 4 \end{pmatrix}$ $P_1(2;-2)$, $Q_1(6;-3)$ and $R_1(7;4)$

A rotation of 90° clockwise about the origin or

A rotation of 270° anticlockwise about the origin

(b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} P & Q & R \\ 2 & 3 & -4 \\ 2 & 6 & 7 \end{pmatrix}=\begin{pmatrix} P_1 & Q_1 & R_1 \\ -2 & -6 & -7 \\ 2 & 3 & -4 \end{pmatrix}$ $P_1(-2;2)$, $Q_1(-6;3)$ and $R_1(-7;-4)$

A rotation of 270° clockwise about the origin

A rotation of 90° anticlockwise about the origin or

Activity (*27.1*) Rotations

Questions

1. On a sheet of graph paper, draw and label Δ Q with vertices X(4;4), Y(1;4) and Z(1;2) and Δ R with vertices $X_1(3;-1)$, $Y_1(3;4)$ and $Z_1(5;4)$ using a suitable scale. Hence, describe fully the transformation that maps Δ Q onto Δ R.
   Hint ;(geometrical constructions can be used where necessary)
2. Find the matrices which are equivalent to clockwise rotations of
   (a) 270° (b) 90°
3. Draw axes such that $-6 \leq x \leq 12$ and $-6 \leq y \leq 12$, using a scale of 1cm to 1 unit. Draw Δ XYZ with vertices X(-1;8), Y(5;4) and Z(-1;1) and $\Delta X^1Y^1Z^1$ with vertices $X^1(4;1)$, $Y^1(0;5)$ and $Z^1(-3;1)$.
   (a) Construct the perpendicular bisector of XX^1 and YY^1.

(b) Mark the centre of rotation C(the point where the two perpendicular bisectors meet)

(c) Check if that is the centre by rotating each point using the compasses.

(d) Join YC and Y^1C, measure YCY^1 and write down the angle of rotation

4. Draw the axes such that $-6 \leq x \leq 12$ and $-6 \leq y \leq 12$, using a scale of 1cm to 1 unit. Draw Δ PQR with vertices P(-4;-2), Q(2;-2) and R(-4;4) and $\Delta P^1Q^1R^1$ with vertices $P^1(4;0)$, $Q^1(4;6)$ and $R^1(-2;0)$.

(a) Construct the perpendicular bisector of PP^1 and QQ^1.

(b) Mark the centre of rotation M(the point where the two perpendicular bisectors meet)

(c) Check if that is the centre by rotating each point using the compasses.

(d) Join PM and P^1M, measure PMP^1 and write down the angle of rotation

Answers

1. On the diagram

2. (a)

Rotation	90° clockwise about the origin (0;0)	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(b)

Rotation	270° anticlockwise about the origin (0;0)	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

3. On the diagram

4. On the diagram

27.4 ENLARGEMENT USING MATRICES

We covered some aspects on enlargement transformation in Unit 23 but we did not cover much on the use matrices

Given that ΔABC with coordinates A (2;3), B(2;1) and C(4;1) is mapped onto $\Delta A_1B_1C_1$ with coordinates $A_1(4;6)$, $B_1(4;2)$ and $C_1(8;2)$ by an enlargement transformation, where the centre of enlargement is (0;0)

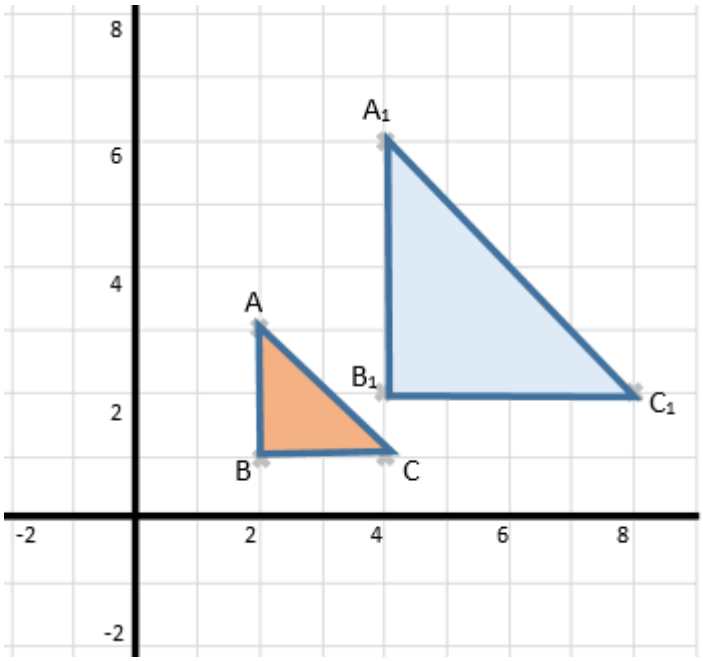


Fig 27.6

With the information given above, a matrix of enlargement at the origin can be derived as follows.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

Four linear equations can be created from the matrices and be solved as follows

$2a + 3b = 4 \ldots . (1)$
$2c + 3d = 6 \ldots . . (2)$
$2a + b = 4 \ldots \ldots (3)$
$2c + d = 2 \ldots \ldots (4)$

By solving equation 1 and 3 simultaneously,

$2a + 3b = 4 \ldots . (1)$
$\underline{-[2a + b = 4 \ldots \ldots (3)]}$
$b = 0$ and
$2a + 3(0) = 4$
$a = 2$

and by solving equations 2 and 4 simultaneously

$2c + 3d = 6 \ldots\ldots (2)$

$\underline{-[2c + d = 2 \ldots\ldots (4)]}$

d = 2 and

2c + (2) = 2

c = 0

Therefore, the matrix of enlargement where the centre is (0;0) is

$E = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2I$

Worked Example [8]

Questions

Triangle OXY has vertices O(0;0), X(3;-6),Y(6;9). OXY is enlarged with scale factor $-\frac{2}{3}$ with the centre (0;0). Find the coordinates of its enlargement $O^1X^1Y^1$.

Solutions

Remember $-\frac{2}{3}\mathbf{I} = -\frac{2}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{pmatrix}$

$$\begin{pmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} O & X & Y \\ 0 & 3 & 6 \\ 0 & -6 & 9 \end{pmatrix} = \begin{pmatrix} O^1 & X^1 & Y^1 \\ 0 & -2 & -4 \\ 0 & 4 & -6 \end{pmatrix}$$

The enlargement has coordinates $O^1(0;0)X^1(-2;4)$and $Y^1(-4;-6)$.

Activity (*27.3*) Enlargement

Questions

Questions

1. Use a suitable scale to draw ΔABC with A(1;1), B(2:1) and C(3:3). Given that ΔABC is enlarge by a scale of -2 and the centre of enlargement is (0;0).
   (a) State the matrix of enlargement
   (b) Draw the image $\Delta A^1B^1C^1$ of ΔABC and label it

2. Use a suitable scale to draw ΔABC with A(8;2), B(6:6) and C(5:3) and $\Delta A^1B^1C^1$ with A^1(6;2), B^1(2;10) and C^1(0;4).

(a) find the centre of enlargement and label it X

(b) find the scale factor of enlargement

(c) derive the matrix of enlargement

3. The matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ represents a transformation **Q** and $\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ represents a transformation **H .** Find

(a) the image of (-3;9) under **Q**

(b) the image of (2;-12) under **H**

(c) the image of (-5;6) under **Q**

(d) image of (-8;14) under **Q**

(e) the image of (6;6) under **H**

(f) image of (-4;-4) under **Q**

(g) the image of (1;15) under **H**

(h) Describe **H** and **Q** as fully as possible

4. Find the matrix **E** which has the effect of enlarging plane shapes by a scale factor -5½ with the origin as centre of enlargement

Answers

1. On the diagram
2. On the diagram
3. (a) (-9;27)

   (b). (-1;6)

   (c) (-15;18)

   (d) (-24;42)

   (e) (-3;-3)

   (f) (-12;-12)

   (g) (-0,5; -7,5)

(h) **H** is an enlargement with a scale factor 3 and centre of enlargement (0;0)

Q is an enlargement with a scale factor -½ and centre of enlargement (0;0)

4. $\begin{pmatrix} -5\frac{1}{2} & 0 \\ 0 & -5\frac{1}{2} \end{pmatrix}$

27.5 STRETCH (ONE WAY AND TWO WAY)

A stretch is a transformation where dimensions are increased or decreased in one or two directions. A one way stretch is an increase or decrease along the x-axis or the y-axis only and a two way stretch is an increase or decrease along both the y-axis and the x-axis

The geometrical representation of a one way stretch

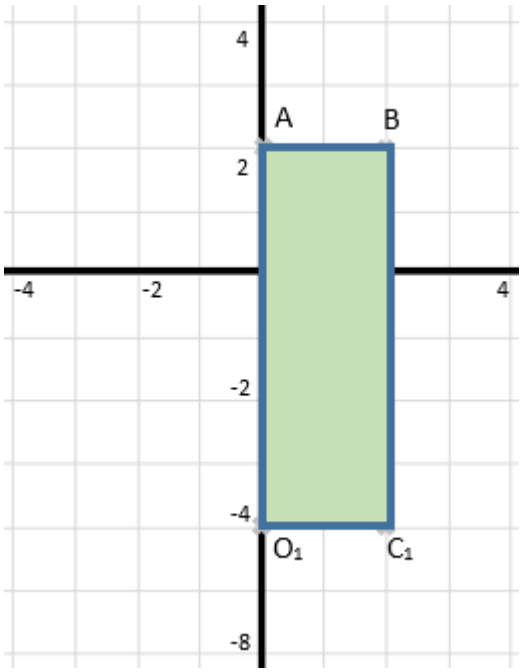


Fig 27.7

-one way stretch along the y-axis

-AB is the in invariant line

-stretch factor = $\frac{BC_1}{BC} = \frac{-6}{2} = -3$

-Area of the image ABO_1C_1 = Area of Original figure × stretch factor

Use of matrices in one way stretches

The stretch matrix where the x- axis is invariant line is $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$

The stretch matrix where the x- axis is invariant line is $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

Where k is the scale factor

Worked Example [9]

Questions

Quadrilateral OABC in figure 25.6 above has vertices O(0;0), A(0;2), B(2;2) and C(2;0). Find

(a) the coordinates of $O_1A_1B_1C_1$ if OABC is stretched using $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

(b) the coordinates of $O_2A_2B_2C_2$ if OABC is stretched using $\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$

Solutions

(a) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} O & A & B & C \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} O_1 & A_1 & B_1 & C_1 \\ 0 & 0 & 6 & 6 \\ 0 & 2 & 2 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} O & A & B & C \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} O_2 & A_2 & B_2 & C_2 \\ 0 & 0 & 2 & 2 \\ 0 & -8 & -8 & 0 \end{pmatrix}$

The stretches have coordinates $O_1(0;0)$, $A_1(0;2)$, $B_1(6;2)$, $C_1(6;0)$ and $O_2(0;0)$, $A_2(0;2)$, $B_2(6;2)$; $C_2(6;0)$

The geometrical representation of a two way stretch

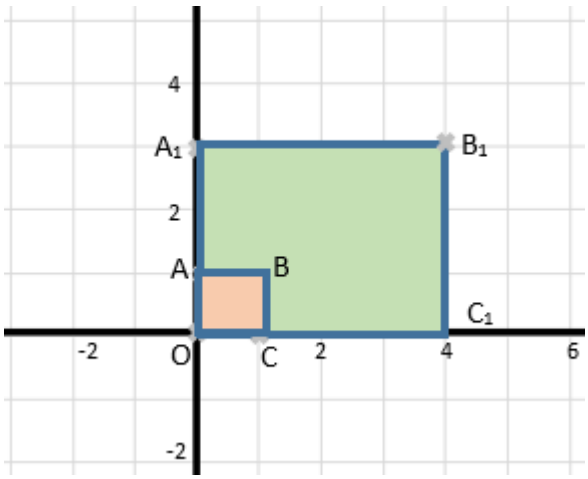


Fig. 27.8

-two way stretch

-O is the in invariant point

-stretch factor = $\frac{OC_1}{OC} = \frac{4}{1} =$ 4 along the x- axis

-stretch factor = $\frac{OA_1}{OA} = \frac{3}{1} =$ 3 along the y- axis

Use of matrices in two way stretches

A two way stretch has the matrix $\begin{pmatrix} k & 0 \\ 0 & h \end{pmatrix}$ when both axes are invariant lines.

Where h and k are stretch factors along the x-axis and the y-axis respectively.

Note: if h = k then the image becomes an enlargement with a scale factor k/h and the centre of enlargement is O(0;0)

Note: By considering Figure 27.8 the matrix representation of the two way stretch is $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ and the inverse of $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ gives the original shape

Worked Example [10]

Questions

Quadrilateral OABC in figure 27.8 above has vertices O(0;0), A(0;2), B(2;2) and C(2;0). Find

(a) the coordinates of $O_1A_1B_1C_1$ if OABC is stretched using $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

(b) the coordinates of $O_2A_2B_2C_2$ if OABC is stretched using $\begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$

Solutions

(a) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} O & A & B & C \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} O_1 & A_1 & B_1 & C_1 \\ 0 & 0 & 6 & 6 \\ 0 & 4 & 4 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} O & A & B & C \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} O_2 & A_2 & B_2 & C_2 \\ 0 & 0 & 4 & 4 \\ 0 & -8 & -8 & 0 \end{pmatrix}$

The stretches have coordinates O_1(0;0), A_1(0;4), B_1(6;4), C_1(6;0) and O_2(0;0), A_2(0;-8), B_2(4;-8)C_2(4;0)

Activity (27.4) Stretches

Questions

1. $X^1(0;0)$, $Y^1(-2;9)$ and $Z^1(-5;6)$ are the images of X(0;0), Y(2;-3) and Z(5;2) and a transformation of represented by a matrix of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find
   (a) the transformation matrix
   (b) the matrix which will transform points X^1, Y^1 and Z^1 back to X,Y and Z
   (c) compare the two matrices and state their relationship
2. The matrix $\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$ represents a transformation **M** and $\begin{pmatrix} -\frac{3}{4} & 0 \\ 0 & 2 \end{pmatrix}$ represents a transformation **N .** Find

(a) the image of (0;10) under **M**
(b) the image of (-1;-2) under **M**
(c) the image of (-6;-5) under **M**
(d) image of (-18;1) under **M**
(e) the image of (16;8) under **N**
(f) image of (-4;-4) under **N**
(g) the image of (12;24) under **N**
(h) Describe **M** and **N** as fully as possible

(i) Given that a transformation maps A(4:0), onto $A_1(4;7)$ and $A_1(4;7)$ is mapped onto $A_2(4;2)$. Determine the matrix representing the transformation

Answers

1.

(a) $\begin{pmatrix} -1 & 0 \\ \frac{36}{19} & \frac{-33}{19} \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 \\ -\frac{12}{11} & -\frac{19}{33} \end{pmatrix}$

(c) The matrix in (a) is the inverse of the matrix in (b)

2.

(a) (0;30)
(b) (2;-6)
(c) (12;-15)
(d) (36;3)
(e) (-12;16)
(f) (3;-8)
(g) (-9;48)

(h) **M** is 2 way stretch matrix with a scale factor -2 along the x-axis and a scale factor -3 along the y – axis and

N is 2 way stretch matrix with a scale factor $-\frac{3}{4}$ along the x-axis and a scale factor 2 along the y – axis and

(i) $\begin{pmatrix} 1 & 0 \\ \frac{7}{4} & -\frac{5}{7} \end{pmatrix}$

27.6 SHEAR USING MATRICES AND GEOMETRICAL METHODS

In Unit 13 we covered the concept on parallelograms and triangles on the same base between the same parallel lines and this concept is a representation of shearing objects.

Geometrical representation of a shear

Below is diagram illustrating a shear of a parallelograms.

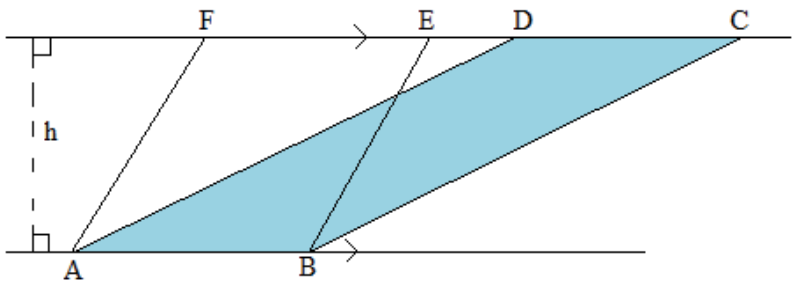


Fig. 27.9

Note:

- The area of parallelogram ABEF = the area of parallelogram ABCD.
- The bottom line AB is the invariant line (a line that does not move or change during the transformation process).
- Points D and C move parallel to the invariant line (one invariant line)
- Shear factor $\frac{\text{Distance moved by a point}}{\text{distance of the point from invarient line}}$

Therefore, a shear is transformation which results in a figure changing shape but its area remains unchanged.

Worked Example [11]

Questions

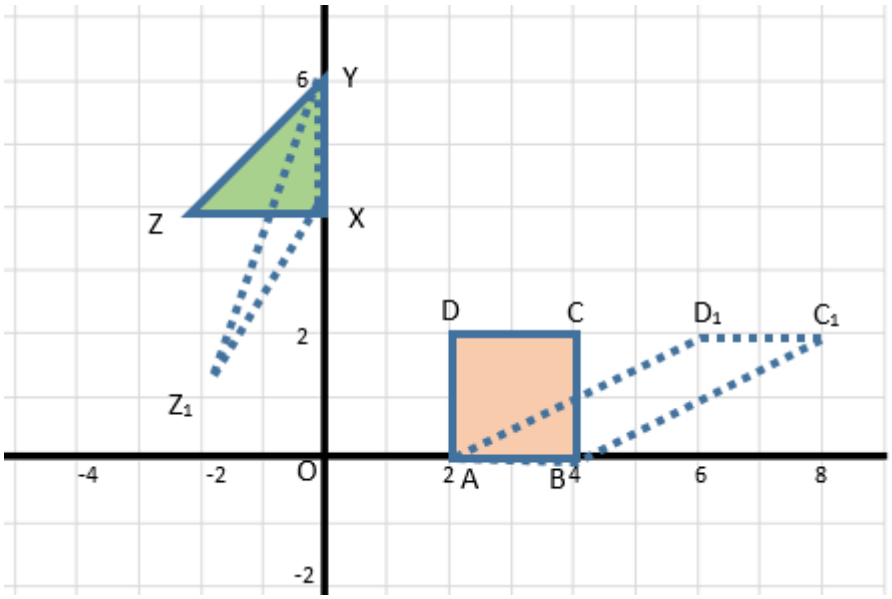


Fig. 27.10

In Figure 27.10 ΔXYZ_1 is an image of ΔXYZ where XY is the invariant line parallel to the y-axis (x=0). We also give that Quadrilateral ABC_1D_1 is an image of ABCD under a shear transformation, x-axis (y=0) invariant. Find the shear factor of each transformation.

Solutions

$$\text{Shear factor of } (XYZ_1) = \frac{\text{Distance moved by Z}}{\text{distance of Z from the invarient line}}$$

$$= \frac{-3}{2} = -1\frac{1}{2}$$

$$\text{Shear factor of } (ABC_1D_1) = \frac{\text{Distance moved by C}}{\text{distance of C from the invarient line}}$$

$$= \frac{4}{2} = 2$$

Shear matrices

The general matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ is the shear matrix where x-axis invariant and $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ is the shear matrix where y-axis invariant and k is the shear factor in each case.

Worked Example [12]

Questions

Shear Triangle ABC, with A(2;2) B(3;4) and C(4;7), by a scale factor -3 with,

(a) y = 0 invariant (b) x = 0 invariant

Solutions

Remember $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

(a) $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B & C \\ 2 & 3 & 4 \\ 2 & 4 & 7 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & C_1 \\ -4 & -9 & -17 \\ 2 & 4 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} A & B & C \\ 2 & 3 & 4 \\ 2 & 4 & 7 \end{pmatrix} = \begin{pmatrix} A_2 & B_2 & C_2 \\ 2 & 3 & 4 \\ -4 & -5 & -5 \end{pmatrix}$

The sheared object has coordinates A_1(-4;2), B_1(-9;4), C_1(-17;7) and A_2(2;-4),B_2(3;-5)C_2(4;-5). This information can be represented on the Cartesian plane as well

Activity (*27.4*) Shear transformations

Questions

1. Using a suitable scale draw triangle A(2;0), B(2;1) and C(0;1). Triangle ABC is transformed by a shear matrix of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to give A_1B_1C where A_1 is (2;2).
   (a) Draw the image B_1. (b) Find the shear factor
2. Using $-4 \le x \le 4$ and $-1 \le y \le 3$ and suitable scale draw Quadrilateral O(0;0) A(0;2.5), B(2;2.5) and C(2;0). Quadrilateral OABC is transformed by a shear to OA_1B_1C where A_1 is (-4; 2.5) and B_1(-2;2.5). Find
   (a) the shear factor (b) the shear matrix
3. The matrix $\begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{pmatrix}$ represents a transformation **A** and $\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$ represents a transformation **B** . Find
   (a) the image of (10;-3) under **B** (b) the image of (-4;-1) under **B**
   (c) the image of (-7; ½) under **B** (d) image of (-12;18) under **A**
   (e) the image of (15;-9) under **A** (f) image of (-3;-3) under **A**

(g) the image of (24;12) under **A**

4. A shear **Q** is represented by the matrix $\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$

   (a) Calculate the coordinates of the image of the point (10; -10) under **Q**

   (b) Calculate the coordinates of the point which will be mapped onto (9;6) by **Q** and write down the equation of the invariant line.

Answers

1. On the diagram
2. On the diagram
3.

   (a) (-8;-3) **(b)** (-10;-1)

   (c) (-4;0,5) **(d)** (-12;22)

   (e) (15;-14) **(f)** (-3;-2)

   (g) (24;4)

4.

   (a) (10;-50) **(b)** (9;42), x = 0

27.7 COMBINATION OF TRANSFORMATIONS

Under this concept we will dwell much on the examination techniques since this part of the unit combines all transformation and this where your examiner is mostly interested on. The examples given below will combine the transformation and give proper notation of combined transformation

Worked Example [13]

Questions

Triangle XYZ has vertices X(2;1), Y(3;1) and Z(3;2). Find the vertices if the triangle is rotated (R) 90° anticlockwise and then enlarged (E) by scale factor 2 with a centre of enlargement (0;0)

Solution

Remember $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

First, rotate, that is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} X & Y & Z \\ 2 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} X_1 & Y_1 & Z_1 \\ -1 & -1 & -2 \\ 2 & 2 & 3 \end{pmatrix}$$

Second, enlarge the points X_1, Y_1, and Z_1, that is

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} X_1 & Y_1 & Z_1 \\ -1 & -1 & -2 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} X_2 & Y_2 & Z_2 \\ -2 & -2 & -4 \\ 4 & 4 & 6 \end{pmatrix}$$

Worked Example [14]

Questions

Triangle OAB has vertices O(0;0), A(2;2) and B(-2;5). Find the vertices if the triangle is sheared (S) by scale factor -2 y axis invariant and then translated T=$\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

Solution

Remember $S = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

First, Shear, that is

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} O & A & B \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} O_1 & A_1 & B_1 \\ 0 & 2 & -2 \\ 0 & -2 & 9 \end{pmatrix}\begin{pmatrix} 0 & 2 & -2 \\ 0 & -2 & 9 \end{pmatrix}$$

Second, translate the points O_1, A_1, and B_1, that is

$$\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}-3\\4\end{pmatrix}=\begin{pmatrix}-3\\4\end{pmatrix}$$

$$\begin{pmatrix}2\\-2\end{pmatrix}+\begin{pmatrix}-3\\4\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix}$$

$$\begin{pmatrix}-2\\9\end{pmatrix}+\begin{pmatrix}-3\\4\end{pmatrix}=\begin{pmatrix}-3\\4\end{pmatrix}$$

The vertices of the translated triangle are $O_2(-3;4)$, $A_2(-1;2)$ and $B_2(-3;4)$

Activity (27.6) Combined transformations

Questions

1. Using suitable scale on a sheet of graph paper show all combined transformation of example 11 in this unit
2. Using suitable scale on a sheet of graph paper show all combined transformation of example 12 in this unit

Answers

1. On the diagram

2. On the diagram

REFLECTION

- All transformations can be represented geometrically or in matrix form
- Matrices of transformation can be derived if the pre-image points and corresponding images points are available

27.8 Summary

This unit has made clear that transformations 1 in unit 23 were incomplete without the use of matrices in transformation. Combined transformations cover the largest part of examination questions on transformation. Assessment questions were

extracted from the past examination questions and this was done to assist you in critical areas of transformations

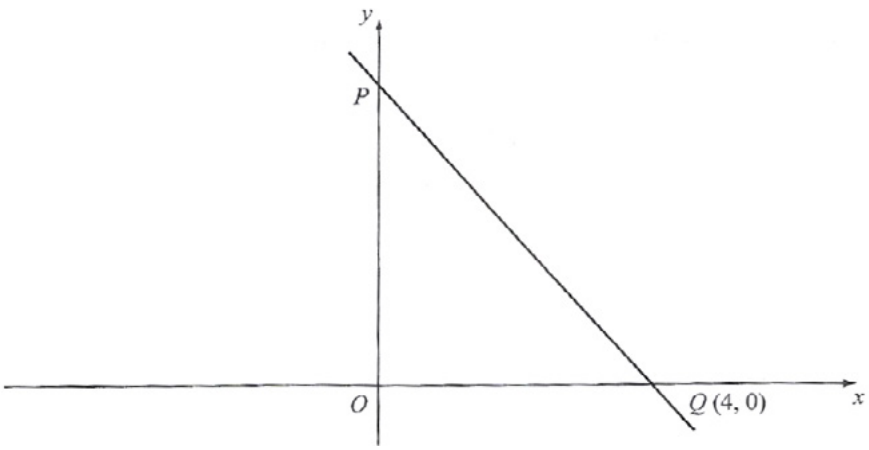
27.9 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

27.10 Assessment Test

1. QN 7 ZIMSEC NOV 1999 P2

(a)



In the diagram, O is the origin. PQ is a straight line with a gradient of -3 and Q is the point $(4, 0)$.

(i) Find the coordinates of P. [1]

(ii) PQ is now reflected in the y-axis to the line PR (not shown), R being the point where the line meets the x-axis.
Find the equation of PR. [3]

(b) $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ is the matrix representing a transformation V and $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ is the matrix representing a transformation W.

(i) Calculate the coordinates of the image of (3, –2) under the transformation VW. [3]

(ii) Find the coordinates of a point whose image under the transformation W is (1, 2). [3]

(iii) Describe completely the transformation V. [2]

2. QN 10 CAMBRIGE NOV 1994 P2

Using a scale of 2 cm to represent 2 units on each axis, draw axes for the following ranges:

$$-6 \leqslant x \leqslant 12 \text{ and } -4 \leqslant y \leqslant 12.$$

(a) ABC is a triangle whose vertices are at the points (2, 6), (2, 8) and (6, 6) respectively. Draw and label this triangle. [1]

(b) Triangle $A_1B_1C_1$ has vertices at the points (–4, 0), (–4, 2) and (0, 0) respectively. Draw and label this triangle.

Triangle ABC can be mapped onto triangle $A_1B_1C_1$ by a **single** transformation. Describe this transformation fully. [3]

(c) Draw the image of triangle ABC under a rotation about the point (7, 2) through 180° and label it $A_2B_2C_2$. [2]

(d) Draw and label the image $A_3B_3C_3$ of triangle ABC under a shear with the x-axis invariant and a scale factor of –1. [3]

(e) Draw and label the image $A_4B_4C_4$ of triangle ABC under a one way stretch with a scale factor of $\frac{3}{2}$ and the x-axis invariant. [3]

3. QN 10b ZIMSEC NOV 2014 REPLACEMENT P2

Use a scale of 2 cm to represent 2 units on x-axis and 2 cm to represent 1 unit on y-axis for $-4 \leq x \leq 14$ and $-5 \leq y \leq 5$.

(i) Triangle ABC has vertices A (4; 1), B (6; 1) and C (6; 2).

Draw and label triangle ABC.

(ii) Transformation **T** represents a translation vector $\begin{pmatrix} -2 \\ -6 \end{pmatrix}$.

Draw and label triangle $A_1B_1C_1$, the image of triangle ABC under **T**.

(iii) Transformation **R** represents a clockwise rotation of 90° about (4; 4).

Draw and label triangle $A_2B_2C_2$, the image of triangle ABC under **R**.

4. QN 12 ZIMSEC NOV 2011 P2

Quadrilateral Q has vertices (–2; 0), (–3; 0), $\left(-3;\ 1\frac{1}{2}\right)$ and (–2; 1).

Using a scale of 2 cm to represent one unit on both axes, draw the x and y-axes for $-6 \le x \le 2$ and $-3 \le y \le 5$.

(a) Draw and label quadrilateral Q. [1]

(b) Quadrilateral Q is mapped onto Q_1 by a reflection in the line $y = 1 - x$.

(i) Draw the line $y = 1 - x$.

(ii) Draw and label quadrilateral Q_1. [4]

(c) Quadrilateral Q is mapped onto quadrilateral Q_2 with vertices (–1; –1), (–1; –2), $\left(-2\frac{1}{2};-2\right)$ and $\left(-2;-1\right)$.

(i) Draw and label quadrilateral Q_2.

(ii) Describe completely, the **single** transformation which maps quadrilateral Q_1 onto Q_2. [4]

(d) Quadrilateral Q_3 is the image of Q under a transformation represented by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Draw and label quadrilateral Q_3. [3]

5. QN 11 ZIMSEC NOV 2000 P2

Triangle X has its vertices at (-4, 4), (-2, 2) and (0, 4). Using a scale of 1cm to represent 1 unit on each axis, draw axes for the values of x and y in the ranges $-4 \le x \le 14$ and $-6 \le y \le 12$ respectively.

(a) Draw and label triangle X. [1]

(b) A single transformation Z maps triangle X onto triangle Z (X) which has vertices at (6, -6), (3, -3) and (0, -6).

(i) Draw and label triangle Z(X).
(ii) Describe fully the transformation Z. [4]

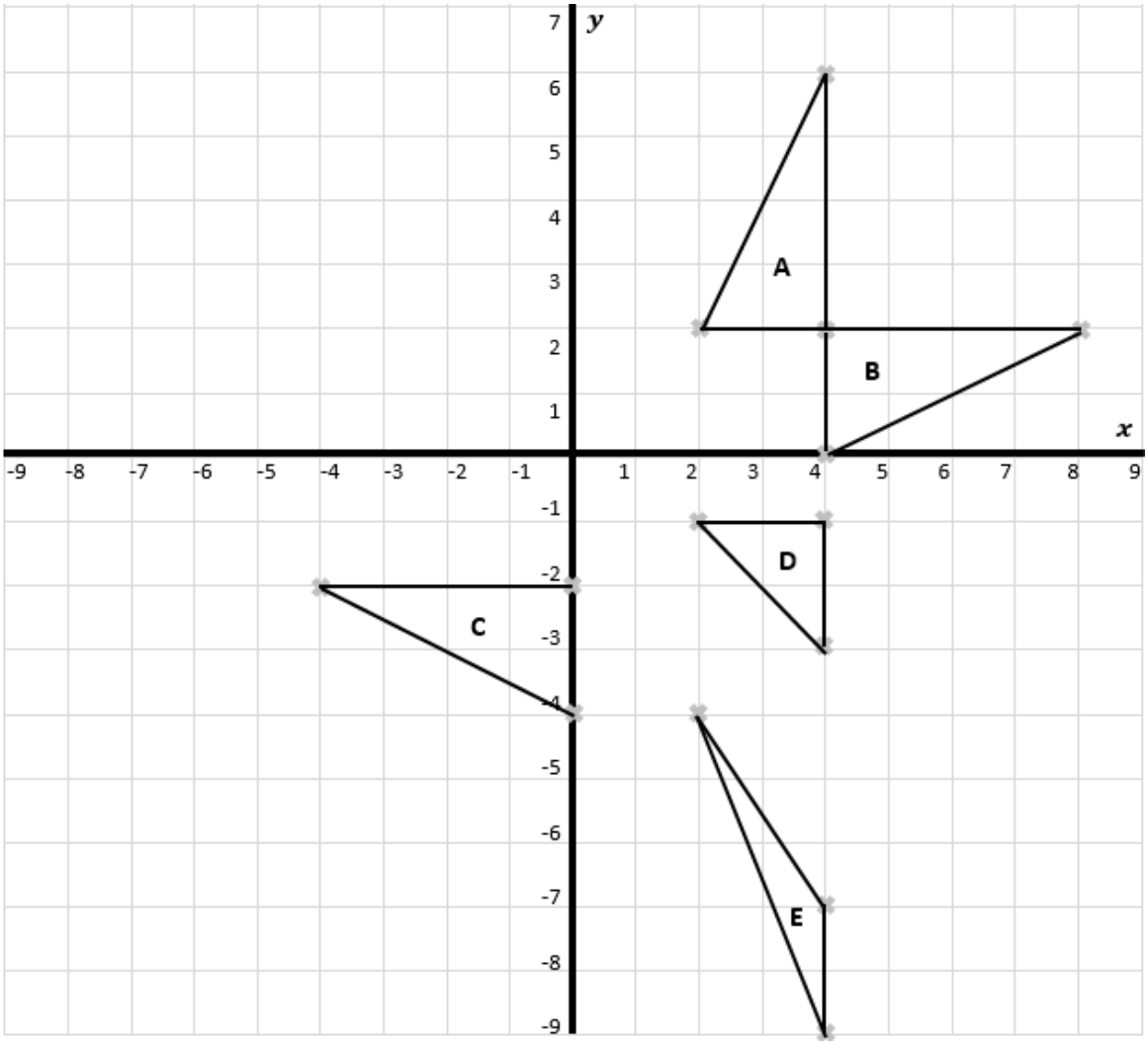
(c) The transformation T is a translation which maps triangle X onto triangle T(X) with the vertex (-4, 4) of triangle X mapped to the point (8,2).

(i) Write down the translation vector T.

(ii) Draw and label triangle T(X). [4]

(d) The transformation R is a clockwise rotation of 90^0 about (4, 2). Draw and label triangle R(X). [3]

6. QN 8 ZIMSEC NOV 2003 P2



Use the graph to answer the following questions.

(a) ΔA is mapped onto ΔB by a reflection in a line l (not shown on the graph).

Write down the equation of line l. [1]

(b) Describe fully a single transformation which maps ΔA onto ΔC. [3]

(c) Write down the coordinates of the image of (4; 6) under an enlargement, centre (0; 2) and scale factor $-\frac{1}{2}$. [2]

(d) ΔA is mapped onto ΔD by a single transformation. Write down the matrix which represents this transformation. [2]

(e) Describe fully a single transformation which maps ΔD onto ΔE. [3]

GLOSSARY

∝ **-** is a variation sign which joins 2 or more quantities which are proportional to each other.

Acceleration- is the rate of change of velocity or speed with time

Acute angle - is an angle which is between 0 and 90 degrees.

Algebra – use of letters of the alphabet to stand for numbers

Angle of depression – is an angle of lowering to the ground level.

Angle of elevation – is an angle of rising from a level ground.

Arc: - Is a part of the circumference or an incomplete circle.

Bank statement: - a bank record of transactions made for the account holder by the bank over, usually a month.

Bearing – is a direction given as the number of degrees

Binary – means base two

Binary – means base two

Bisection - this is to divide into two equal pieces or parts

Chord: - Is a line joining two points on a circle but not passing through the centre of that circle.

Commission: - payment given to the worker to encourage hardworking.

Compound interest: - non fixed interest paid over a changing principal.

Construction - is the drawing of shapes, angles and lines accurately using geometrical or mathematical instruments

Continuous data - This is information that does not take exact values for example age or weight of people in a family

Convex: - Closed.

Cosine angle - is a ratio that is defined by adjacent side over the hypotenuse side of a right angled triangle.

Cumulative frequency – the total number of frequencies in a frequency distribution.

Data – Refers to the facts and figures collected together for the purpose of analysis

Deceleration-is the decrease in velocity/speed with time

Denary – means base ten

Deposit: - money banked.

Depreciation: - this is reduction in goods value over a period of time.

Determinant - determines whether the matrix is invertible or not

Diameter: - Is a line joining two points on the circumference passing through the centre of that circle.

Direct variation – is the variation which includes 2 quantities varying directly to each other.

Discount: - it is the money deducted from marked price when you pay cash.

Discrete data – Refers to data that can only be represented by whole numbers for example number of people admitted at a hospital

Displacement – distance or length covered by a moving object in a specific direction

Distance- any length covered by a moving object

Equation- is an algebraic statement that contains an equal sign.

Equiangular – refers to similar triangles. Equal in angle sizes, but different in size of the shapes

Equidistant - the word is made up of two words, equal and distance therefore the word means being at an equal distant from the same point or thing.
being at an equal distant from the same point or thing.

Event - a set of outcomes of an experiment to which a probability is assigned, that is, it is the result of an experiment

Expand – remove brackets from an expression

Experiment – It is any activity with an observable result, that is, a well-defined set of possible outcomes.

Exposed surface area – the size of a solid shape that can be seen from outside

Expression – a statement without an equal sign

Factorise – writing an expression in terms of its factors

Formulae - A statement, especially an equation used to express a mathematical rule or principle.

Frequency – It is the number of times a data value occurs, that is, the number of occurrences of a particular item in a set of data

Frustum – may be formed from a right circular cone or a pyramid by cutting off the tip of the cone or pyramid with a cut perpendicular to the height, forming a lower base and an upper base that are circular and parallel.

Gradient - this is the measure of the steepness of a line.

Gross income – is one's salary before deductions.

Hire purchase (HP): - payment spread over agreed period of time for an item which you cannot pay for cash.

Hypotenuse side – is the longest side of a right-angled triangle and is opposite the 90 degree angle.

Included angle – an angle between two given sides of a triangle

Index – is the number in the form x^a where x is the base and a is the power. The plural for index is indices.

Instalments: - part payment made to cover hire purchase.

Integers – these are positive and negative whole numbers.

Interest: - is the money you pay back when you borrow or is the money you get a after saving for a certain period.

Intersection- is the point where two lines or curves meet or join.

Inverse variation –is the variation which involves 2 quantities that vary indirectly to each other that is as one quantity increases the other decreases and vice versa.

Irrational numbers – these are numbers that cannot be written as fractions

Isometric – the shape produced after transformation maintains the same dimensions as the original shape, that is, shape remains congruent to the original shape after transformation.

Joint variation- is the variation which involves 3 or more quantities joined together with a multiplicative rule.

Line of symmetry – the line which divide a shape (a curve in our case) into two equal parts

Linear equation – It is an algebraic equation in which each term has an exponent of one.

Linear equation – It is an algebraic equation in which each term has an exponent of one.

Locus - is a set of points satisfying a given rule

Logarithm - a logarithm is a power

Matrix – is the rectangular arrangement of elements or objects

Maximum- this is the highest or greatest point reached

Mean – it is the average of a set of numbers

Median – It is a simple measure of central tendency, that is, the middle value in a set of numbers arranged in rank order

Minimum - this is the least value or the smallest value reached

Mode – It is the most frequently occurring number found in a set of numbers

Net income – is also called take home. It is the pay of an individual after deductions.

Non-Isometric – the shape produced after transformation does not maintain the same dimensions as the original shape, that is, it changes shape after transformation.

Obtuse angle – is an angle greater than 90 but less than 180 degrees.

Order of a matrix – is the number of rows and columns of a given matrix

Outcome – It is a possible result of an experiment

Partial variation- is the variation that occurs when one quantity has a fixed value and at the same time it is also proportional to another variable.

PAYE –is the tax which is deducted from salaries.

Polygon: - Is any plane figure with straight edges or sides.

Possible outcomes – These are the possible results of an experiment

Principal: - money borrowed or saved.

Prism – a solid shape that has the same cross-section all along the shape from end to end.

Probability is the study of chance or the likelihood of an event happening.

Pythagoras – was a famous Greek Mathematician (philosopher) who discovered the rule or formula we call Pythagoras Theorem.

Quadractic Expression – an expression where highest power of the unknown is 2.

Quartiles – These are the values that divide data into quarters

Random event - A random event is something unpredictable hence you can never give it an exact value or probability.

Range – It is the difference between the lowest and highest values in a given set of data

Rational numbers – these are numbers that can be written as fractions

Regular: - Same length and angles are same size.

Required outcomes –that is what has actually happened.

Sample space – It is set of all possible outcomes of an experiment. The sample space for the experiment of flipping a coin is heads and tails.

Scalar – a quantity that has size only or a numerical value that multiplies, reduces or stretches a vector or a matrix

Segments: - Are two parts of the circle cut by a chord.

Sequence – a set of numbers with well-defined order or pattern

Set builder notation – It is a shorthand method for describing sets and it involves use of symbols to describe elements of a set as well as their properties.

Set builder notation: - It is a shorthand method for describing sets and it involves use of symbols to describe elements of a set as well as their properties.

Similar – objects are similar if they show equivalency in characteristics except for size

Singular matrix - is a matrix which has a determinant of zero, it has no inverse.

Solve- Find the value of the unknown

Standard form – this involves writing a number in the form $A \times 10^n$ where A is a number between 1 and 10 while n is a positive or negative whole number

Statistics - It is a form of mathematical analysis concerned with collecting, organizing, and interpreting data

Subject of the formulae – It is a variable which is expressed in terms of other variables involved in the formula.

Substitute – to replace a letter with a number

Subtend: - Is an angle formed by two lines meeting at a point on the circumference from each end of a chord.

Supplementary angles – these are two angles which add to 180°.

Surface area – the size of the surface of a shape, and that area is measured in square units

Symmetry: - A line which divide a shape into two equal parts or mirror line.

Tangent – this is a straight line which passes through a point on a curve, it just touches a curve at a point

Theorem: - Is a proven law or fact or rule.

Translation – it is a vector quantity which involves movement in a straight line and without turning

Transversal line: - Line which crosses or cut parallel lines.

Unknown – A letter in an equation, representing a number.

Variables – letters that are used to represent numbers.

VAT–is the tax levied on sales of goods and services

Vector – a quantity with direction and size (magnitude or modulus) e.g. velocity, acceleration, displacement, force as well as translation

Velocity – is the displacement per unit time

Venn diagrams – This is another way of presenting sets using diagrams a rectangle to represent universal set and circles to represent subsets.

Vertex - the point of intersection of any lines on a given shape. It might be at times called corners of a shape.

Vertex: - Point where two lines of a triangle meet, vertices when they are many.

Volume – is the number of cubic units that make up a solid figure.

Withdrawal: - money taken from the bank.